6. [14 points] Consider the power series

$$\sum_{n=3}^{\infty} \frac{(3-x)^{3n}}{8^n(n-2)}.$$

(a) [5 points of 14] Does this power series converge at $x = 1$? Explain.

Solution:
No. When $x = 1$, the series becomes $\sum_{n=3}^{\infty} \frac{2^{3n}}{8^n(n-2)} = \sum_{n=3}^{\infty} \frac{1}{(n-2)}$. Then $\frac{1}{n-2} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges, so this must diverge also.

(b) [5 points of 14] Does this power series converge at $x = 5$? Explain.

Solution:
Yes. When $x = 5$, the series becomes $\sum_{n=3}^{\infty} \frac{(-2)^{3n}}{8^n(n-2)} = \sum_{n=3}^{\infty} \frac{(-1)^{3n}}{(n-2)}$. The terms of this series alternate sign, are decreasing in magnitude, and $\lim_{n\to\infty} \frac{1}{n-2} = 0$, so by the alternating series test the series converges.

(c) [4 points of 14] Find the interval of convergence of this power series. Be sure it is clear how you find your answer.

Solution:
To find the radius of convergence we use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(3-x)^{3n+3}}{8^{n+1}(n-1)} \cdot \frac{8^n(n-2)}{(3-x)^{3n}}\right| = |(3-x)^3| \lim_{n\to\infty} \frac{n-2}{8(n-1)} = \frac{1}{8}|(3-x)^3|.$$

For convergence, we know that this ratio must be less than one, so $|x-3|^3 < 8$, or $|x-3| < 2$, and the radius of convergence is 2. Thus the interval of convergence is at least $1 < x < 5$, and, from the work in (a) and (b), we know that it in fact includes $x = 5$ but not $x = 1$. The interval of convergence is $1 < x \leq 5$.

Alternately, we know that the series is centered on $x = 3$. At $x = 1$ and $x = 5$ (a distance of two from $x = 3$) it diverges and converges, respectively. Thus the interval of convergence is $1 < x \leq 5$.