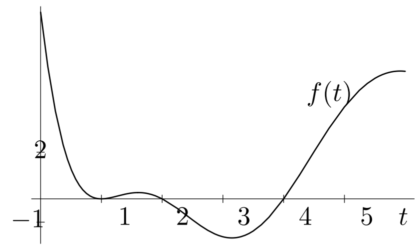
7. [10 points] Define a function F for $x \geq 0$ by $F(x) = \int_x^{2x} f(t)\,dt$, where $f(t)$ is given by the graph to the right.



(a) [4 points of 10] Find $F'(1)$ (show your work).

Solution:
We know that $F(x) = \int_x^a f(t)\,dt + \int_a^{2x} f(t)\,dt = -\int_a^x f(t)\,dt + \int_a^{2x} f(t)\,dt$, for some a between x and $2x$. Thus, from the construction theorem (second fundamental theorem of calculus), $F'(x) = -f(x) + 2f(2x)$, and we therefore have $F'(1) = -f(1) + 2f(2) = 0$.

(b) [6 points of 10] If the second degree Taylor polynomial for $F(x)$ near $x = 1$ is $P_2(x) = a + b(x-1) + c(x-1)^2$, what is b? What is the sign of a? The sign of c? Why?

Solution:
We know that $a = F(1)$, $b = F'(1)$ and $c = \frac{1}{2}F''(1)$. Thus, from (a), we know that $b = 0$. Then $a = \int_1^2 f(t)\,dt$. The area under $f(t)$ is above the t-axis between $t = 1$ and $t = 2$, so $a > 0$. Finally, $F''(x) = \frac{d}{dx}(-f(x) + 2f(2x)) = -f'(x) + 4f'(2x)$, so $F''(1) = -f'(1) + 4f'(2) = 0 + 4f'(2)$. At $x = 2$, the slope of f is negative, so $F''(1) < 0$ and similarly $c < 0$.