8. [10 points] The function $f(x) = \dfrac{c}{1+x^2}$ $(-\infty < x < \infty)$ occurs in probability theory as the density function of the Cauchy distribution.

(a) [5 points of 10] Find c (show your work).

Solution:
For this to be a density function, we know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. This $f(x)$ is symmetric about $x = 0$, so this is equivalent to taking half of the integral to be $1/2$.

$$\int_0^\infty \frac{c}{1+x^2}\,dx = \lim_{b\to\infty} c\cdot\arctan(b) - c\cdot\arctan(0) = \lim_{b\to\infty} c\cdot\arctan(b) = \frac{\pi}{2}c.$$

Thus $\frac{\pi}{2}c = \frac{1}{2}$, and $c = \frac{1}{\pi}(\approx 0.3183)$.

(b) [5 points of 10] Electrostatic charge is distributed along an infinite straight rod according to the Cauchy distribution. What proportion of the charge is located more than one unit of length away from the origin?

Solution:
This is equal to the area under $f(x)$ for $|x| > 1$, or, equivalently, $1 - \int_{-1}^{1} f(x)\,dx$. Thus

$$\text{proportion} = 1 - \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 1 - \frac{2}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx = 1 - \frac{2}{\pi}\arctan(1) = 1 - \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}.$$