4. [12 points] The density of the Earth changes with the distance below the surface of the Earth one goes. If x gives the distance (in km) below the surface, the density $\delta(x)$ (in kg/km^3) is approximately

x	0	1000	2000	2900	3000	4000	5000	6000	6370
$\delta(x)$	3300	4500	5100	5600	10,100	11,400	12,600	13,000	13,000

(the radius R of the Earth is about 6370 km). Let r measure the distance out from the center of the Earth.

a. [4 points] The integral $\int_0^{4000} (4\pi \cdot r^2 \cdot \delta(R-r))\, dr$ is the limit as $\Delta r \to 0$ of a Riemann sum $\sum 4\pi \cdot r^2 \cdot \delta(R-r) \cdot \Delta r$. In the context of this problem, what do the terms of this sum represent?

Solution: The terms of the sum are "slices" of mass at a radius r with a thickness Δr. The expression $4\pi\, r^2\, \Delta r$ is the volume of a spherical shell with a radius r, and $\delta(R-r)$ is the density at that radius.

b. [4 points] Now consider the integral $\int_{R-4000}^{R} (4\pi \cdot r^2 \cdot \delta(R-r))\, dr$. Rewrite this in terms the variable x. Estimate your rewritten integral with MID(2).

Solution: We know r is the radial distance out from the center of the earth, so $x = R-r$, which gives $dx = -dr$, and we can substitute to rewrite the integral:

$$\int_{R-4000}^{R} 4\pi \cdot r^2 \cdot \delta(R-r)\, dr = -\int_{4000}^{0} 4\pi \cdot (R-x)^2 \cdot \delta(x)\, dx = \int_{0}^{4000} 4\pi \cdot (R-x)^2 \cdot \delta(x)\, dx.$$

Estimating this with MID(2), we have

$$\begin{aligned}\int_0^{4000} 4\pi \cdot (R-x)^2 \cdot \delta(x)\, dx &\approx 4\pi \cdot (2000)\left((R-1000)^2\,(4500) + (R-3000)^2\,(10,100)\right)\\ &\approx 6.144 \times 10^{15} \text{ kg}.\end{aligned}$$

c. [4 points] Let $F(x) = \int_{R-x}^{R} (4\pi \cdot r^2 \cdot \delta(R-r))\, dr$. Find $F'(x)$, showing work that shows how you obtained your answer.

Solution: Using the Second Fundamental Theorem of Calculus,

$$F'(x) = -\frac{d}{dx}\int_R^{R-x} 4\pi\, r^2\, \delta(R-r)\, dr = -(-1)4\pi\,(R-x)^2\,\delta(x) = 4\pi\,(R-x)^2\,\delta(x).$$