5. [12 points] For each of the following series, carefully prove its convergence or divergence. You must clearly indicate what test(s) you use in your proof, and must carefully show all work that demonstrates their appropriateness and the calculations associated with the tests.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 1}{e^n - n}$

Solution: There are a couple of possible methods we could use to show that this series converges. Using the ratio test, we look at $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. This is

$$\lim_{n\to\infty} \left|\frac{2^{n+1}-1}{e^{n+1}-(n+1)} \cdot \frac{e^n - n}{2^n - 1}\right| = \lim_{n\to\infty} \left|\frac{e^n 2^n (2 - \frac{1}{2^n})(1 - \frac{n}{e^n})}{e^n 2^n (e - \frac{n+1}{e^n})(1 - \frac{1}{2^n})}\right| = \frac{2}{e}.$$

The limit is less than one, so by the ratio test we know that this series converges. Alternately, we could use the limit comparison test and compare with the geometric series $\sum \left(\frac{2}{e}\right)^n$; $\frac{2}{e} < 1$, so this is a convergent geometric series. Then we look at $\lim_{n\to\infty} \frac{b_n}{a_n}$, which is

$$\lim_{n\to\infty} \frac{2^n - 1}{e^n - n} \cdot \frac{e^n}{2^n} = \lim_{n\to\infty} \frac{2^n e^n (1 - \frac{1}{e^n})}{2^n e^n (1 - \frac{n}{e^n})} = 1.$$

This is finite, so the convergent properties of the two are the same, and thus our series converges.

b. [6 points] $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^3 + \cos(n)}$

Solution: Note that $a_n = \frac{n}{n^3+\cos(n)} \le \frac{n}{n^3-1} < \frac{n}{n^3 - \frac{1}{4}n^3} = \frac{4n}{3n^3} = \frac{4}{3n^2}$ for $n \ge 2$. Further $a_n > 0$. Thus, by comparison with the series $\frac{4}{3}\sum \frac{1}{n^2}$, which we know converges, we must have that this series converges.

Alternately, we could also use limit comparison with $\sum \frac{1}{n^2}$, which we know converges. The terms of the given and comparison series are positive, so we can use limit comparison. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3 + \cos(n)} = \lim_{n\to\infty} \frac{1}{1 + \frac{\cos(n)}{n^3}} = 1.$$

Thus, by the limit comparison test we know that $\sum \frac{n}{n^3+\cos(n)}$ must converge.