6. [12 points] When a patient takes a drug (e.g., by ingesting a pill), the amount of the drug in her/his system changes with time. We can think of this process discretely (each pill is an immediately delivered dose) or continuously (each pill delivers a small amount of drug per unit time over a long time). This problem considers these two different models.

a. [4 points] Suppose that ibuprofen is taken in 200 mg doses every six hours, and that all 200 mg are delivered to the patient's body immediately when the pill is taken. After six hours, 12.5% of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the nth pill taken. Include work; without work, you may receive no credit.

Solution: Immediately after the first pill, the amount in the patient's system is $A_1 = 200$. Immediately before the second pill is taken the amount in the patient's system is $B_2 = (0.125)(200)$, and immediately after, $A_2 = 200 + (0.125)(200)$. Similarly, we have $B_3 = (0.125)(200) + (0.125)^2(200)$ and $A_3 = 200 + (0.125)(200) + (0.125)^2(200)$, and so on. Thus immediately after the nth pill taken, the patient has

$$A_n = 200 + (0.125)(200) + \cdots + (0.125)^{n-1}(200) = \frac{200(1 - 0.125^n)}{1 - 0.125} \text{ mg},$$

or about $228.57(1 - 0.125^n)$ mg of ibuprofen in her/his system, and immediately before the nth pill the patient has (for $n > 1$),

$$B_n = (0.125)(200) + \cdots + (0.125)^{n-1}(200) = \frac{(0.125)(200)(1 - 0.125^{n-1})}{1 - 0.125} \text{ mg},$$

or about $28.571(1 - 0.125^{n-1})$ mg of ibuprofen in her/his system.

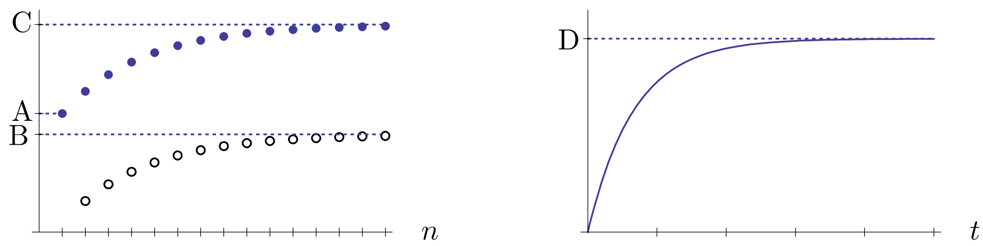
b. [4 points] Now suppose that ibuprofen is taken in a time-release capsule that continuously releases 35 mg/hr of ibuprofen per hour for six hours. The drug decays at a rate proportional to the amount in the body, with a constant of proportionality $r = 0.35$. Write a differential equation for the amount of ibuprofen, $y(t)$, in the patient as a function of time. Solve your differential equation, assuming that there is no ibuprofen in the patient initially.

Solution: We know that the rate of change of the ibuprofen in the body, $y(t)$, is 35 mg/hr less the decay of the amount that's present, so we have $\frac{dy}{dt} = 35 - 0.35y$, and if the initial amount present is zero, we also have $y(0) = 0$. We can solve this by separation of variables:

$$\frac{dy}{dt} = 35 - 0.35y = -0.35(y - 100), \quad \text{so} \quad \frac{dy}{y - 100} = -0.35\,dt.$$

Integrating both sides, we have $\ln|y - 100| = -0.35t + C$, so that $y = 100 + k\,e^{-0.35t}$. Because $y(0) = 0$, $k = -100$, and we have $y = 100(1 - e^{-0.35t})$. Note that this is only valid for $0 \le t \le 6$; at $t = 6$ we expect that another pill will be taken, and we will then have to solve the same differential equation with the initial condition $y(6) = 100(1 - e^{-0.35(6)}) = 87.75$.

c. [4 points] *This is the third part of problem (6), which begins on the previous page.* The figure below shows graphs that may correspond to the solutions that you obtained in (6a) and (6b). The graph to the left corresponds to part (a), with the solid and open circles in the graph showing the amount of ibuprofen in the patient after and before the nth pill. The graph to the right corresponds to (b). Assuming that these graphs do represent your solutions, use your results from (a) and (b) to determine the values of A, B, C, and D.



Solution: We note that A is the amount of ibuprofen in the patient after taking one pill, which is 200 mg; C is the limiting value for this quantity, which is 228.6 mg. B is the limiting value for the amount of ibuprofen in the patient before the nth pill, which is 28.6 mg. D is the limit of the exponential in (b), which is 100 mg.

7. [16 points] The annual snowfall in Ann Arbor has mean 52.91 inches and standard deviation 12.67 inches. Assume that these data are normally distributed, and let x be the deviation of a year's snowfall from the mean (so that if $x = -2$ in a given year, it means that the snowfall that year was 50.91 inches). Then the probability density function for x is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/(2\sigma^2)} = 0.03149\, e^{-x^2/321.1},$$

so that its cumulative distribution function $P(x)$ is

$$P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{x} 0.03149\, e^{-t^2/321.1}\, dt.$$

a. [2 points] Explain why $P(x) = \frac{1}{2} + \int_0^x 0.03149\, e^{-t^2/321.1}\, dt.$

Solution: We know the area under $p(x)$ is one, and because the normal distribution is symmetric about its mean, we know $\int_{-\infty}^{0} p(x)\, dx = \frac{1}{2}$. Thus $P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{0} p(t)\, dt + \int_0^x p(t)\, dt = \frac{1}{2} + \int_0^x p(t)\, dt.$