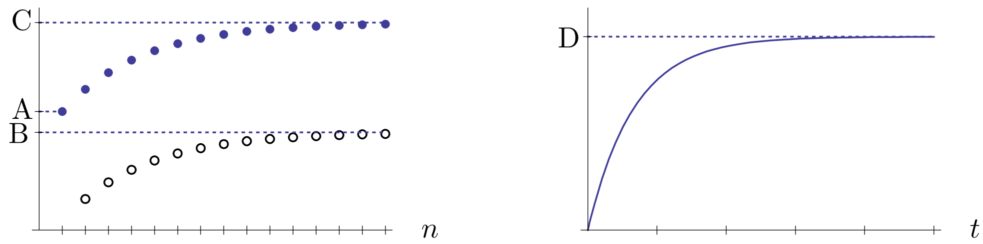
c. [4 points] *This is the third part of problem (6), which begins on the previous page.* The figure below shows graphs that may correspond to the solutions that you obtained in (6a) and (6b). The graph to the left corresponds to part (a), with the solid and open circles in the graph showing the amount of ibuprofen in the patient after and before the nth pill. The graph to the right corresponds to (b). Assuming that these graphs do represent your solutions, use your results from (a) and (b) to determine the values of A, B, C, and D.



Solution: We note that A is the amount of ibuprofen in the patient after taking one pill, which is 200 mg; C is the limiting value for this quantity, which is 228.6 mg. B is the limiting value for the amount of ibuprofen in the patient before the nth pill, which is 28.6 mg. D is the limit of the exponential in (b), which is 100 mg.

7. [16 points] The annual snowfall in Ann Arbor has mean 52.91 inches and standard deviation 12.67 inches. Assume that these data are normally distributed, and let x be the deviation of a year's snowfall from the mean (so that if $x = -2$ in a given year, it means that the snowfall that year was 50.91 inches). Then the probability density function for x is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/(2\sigma^2)} = 0.03149\, e^{-x^2/321.1},$$

so that its cumulative distribution function $P(x)$ is

$$P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{x} 0.03149\, e^{-t^2/321.1}\, dt.$$

a. [2 points] Explain why $P(x) = \dfrac{1}{2} + \displaystyle\int_0^x 0.03149\, e^{-t^2/321.1}\, dt$.

Solution: We know the area under $p(x)$ is one, and because the normal distribution is symmetric about its mean, we know $\int_{-\infty}^{0} p(x)\, dx = \frac{1}{2}$. Thus $P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{0} p(t)\, dt + \int_0^x p(t)\, dt = \frac{1}{2} + \int_0^x p(t)\, dt$.

This continues problem 7: here, $p(x) = 0.03149\, e^{-x^2/321.1}$, *and*

$$P(x) = \frac{1}{2} + \int_0^x 0.03149\, e^{-t^2/321.1}\, dt.$$

b. [5 points] Write a Taylor series for $p(x)$ (around $x = 0$).

Solution: We know that $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$. Thus,

$$\begin{aligned} p(x) &= 0.03149\, e^{-x^2/321.1} \\ &= 0.03149 \left(1 - \frac{1}{321.1}x^2 + \frac{1}{2!\,(321.1^2)}x^4 + \cdots + \frac{(-1)^n}{n!\,(321.1^n)}x^{2n} + \cdots\right). \end{aligned}$$

c. [5 points] Write a Taylor series for $P(x)$ (around $x = 0$). *Hint: you will probably want to use your work from (b).*

Solution: Using the result from (a) and integrating the series in (b), we have

$$\begin{aligned} P(x) &= \frac{1}{2} + \int_0^x 0.03149 \sum_{n=0}^{\infty} \frac{t^{2n}}{n!\, 321.1^n}\, dt \\ &= \frac{1}{2} + 0.03149 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)\, 321.1^n}. \end{aligned}$$

d. [4 points] Use your result from (b) or (c) to estimate the probability that there will be up to 60 inches of snow this winter.

Solution: This probability is just $P(60 - 52.91) = P(7.09)$. Using the first two terms of the series in (c), we have $P(7.09) \approx \frac{1}{2} + 0.03149\left(7.09 - \frac{(7.09)^3}{3\,(321.1)}\right) = 0.712$, or about a 71.2% chance. (The leading term gives 72.3%.)
We could also use the series for $p(x)$: $p(x) \approx 0.03149(1 - \frac{1}{321.1}x^2)$, so the probability we want is $\approx \frac{1}{2} + \int_0^{7.09} 0.03149(1 - \frac{1}{321.1}x^2)\, dx$, which will clearly be the same as the preceding. Obviously, trying to use $\int_{-\infty}^{7.09} 0.03149(1 - \frac{1}{321.1}x^2)\, dx$ does not work.