3. [6 points] Suppose $f(x)$ is a twice-differentiable function. On the interval $[a, b]$, for $0 < a < b$, $f(x)$ is positive, increasing, and concave up. Suppose $g(x) = x(f(x))^2$. If one uses the midpoint rule to estimate $\int_a^b g(x)dx$, will the estimation be an overestimate or an underestimate? Be sure to justify your answer and show all appropriate work. *(Hint: You might find it helpful to consider the concavity of function $g(x)$.)*

Solution: The midpoint rule is an overestimate when the integrand function is concave down, and an underestimate when the integrand function is concave up. In order to determine the concavity of $g(x)$ on the interval $[a, b]$, we need to find $g''(x)$.

$$\begin{aligned} g'(x) &= 2xf(x) \cdot f'(x) + (f(x))^2 \\ g''(x) &= 2x(f(x)f"(x) + (f'(x))^2) + 2f(x)f'(x) + 2f(x) \cdot f'(x) \\ &= 2xf(x)f"(x) + 2x(f'(x))^2 + 4f(x)f'(x) \end{aligned}$$

Since all terms are positive on the interval $[a, b]$, $g(x)$ is concave up, so the estimation will be an underestimate.