5. [14 points] A water filtration system has a rectangular basin 0.2 m wide, 0.4 m long and 0.2 m deep in which unfiltered water is poured. Along the bottom of the basin there is a filter, in the shape of a square, measuring 0.01 m on each side.

a. [3 points] The density of water is 1000 kg/m^3, and the gravitational constant is 9.8 m/sec^2. Suppose the depth of the water in the basin is h m. What is the force due to water pressure on the filter? Include units in your answer.

Solution: Since the filter is on the bottom of the basin, at constant depth, the pressure is $P = \text{density} \cdot \text{gravity} \cdot \text{depth} = (1000kg/m^3)(9.8m/s^2)(hm) = 9800kg/(m \cdot s^2) = 9800Pa$. The force is the pressure on the total area, so $F = P \cdot \text{area} = (9800)(.0001) = 0.98N$.

b. [3 points] Write an equation for the volume of water in the basin, V, when the water is h m deep. Then use this equation to write an equation for the rate of change of volume of water, $\frac{dV}{dt}$, in terms of the rate of change of depth of water, $\frac{dh}{dt}$.

Solution: $V = (0.2)(0.4)h = 0.08h$, so $\frac{dV}{dt} = 0.08\frac{dh}{dt}$

c. [5 points] The rate at which water passes out of the basin is proportional to the square of the force due to water pressure exerted on the filter, with constant of proportionality $k > 0$. Suppose the basin is filled with water at a constant rate r m^3/sec. Write a differential equation for the depth h of the water in the basin. That is, find an equation for $\frac{dh}{dt}$ in terms of h and constants r and k.

Solution: Letting V be the volume of water in the tank, we have

$$\text{rate of change of volume} = \text{rate in} - \text{rate out}.$$

The rate of change into the basin is r, and the rate of flow out is given to be proportional to the square of the force due to pressure in the basin found in part (a) with proportionality constant k. Therefore, the differential equation for the volume is $\frac{dV}{dt} = r - k(0.98h)^2$. Since $\frac{dV}{dt} = 0.08\frac{dh}{dt}$, the differential equation for h is therefore $0.08\frac{dh}{dt} = r - 0.9604kh^2$, so

$$\frac{dh}{dt} = 12.5r - 12.005kh^2$$

d. [3 points] Suppose when the basin is filled at a constant rate of 0.0002 m^3/sec, the depth remains constant at 0.1 m. What is the constant of proportionality k?

Solution: This is saying that the equation above has an equilibrium solution $h = 0.1$ when $r = 0.0002$. Using the differential equation, this means that $0 = 12.5(0.0002) - 12.005k(0.1)^2$, so $k \approx 0.020825$.