7. [14 points] Consider the function $g(x) = \int_0^{x^2} e^{-t^2} dt$.

a. [4 points] Suppose $f(x) = g'(x)$. Find a formula for $f(x)$.

Solution:

$$f(x) = e^{-(x^2)^2} \cdot (2x) = 2xe^{-x^4}$$

b. [6 points] Find the Taylor series of $f(x)$ about $x = 0$. Write your answer using summation (sigma) notation, including proper indices.

Solution:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{-x^4} &= \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \\ f(x) &= 2x \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2x^{4n+1}}{n!} \end{aligned}$$

c. [4 points] Find the Taylor series of $g(x)$ about $x = 0$. Write your answer using summation (sigma) notation, including proper indices.

Solution: Starting with the Taylor series for e^x we have

$$\begin{aligned} g(x) &= \int g'(x) dx \\ &= \int f(x) dx = \sum_{n=0}^{\infty} (-1)^n \frac{2x^{4n+2}}{(4n+2)n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)n!} \end{aligned}$$