7. [14 points] Consider the function $g(x) = \int_0^x e^{-t^2} dt$.

a. [4 points] Suppose $f(x) = g'(x)$. Find a formula for $f(x)$.

Solution:

$$f(x) = e^{-(x^2)^2} \cdot (2x) = 2xe^{-x^4}$$

b. [6 points] Find the Taylor series of $f(x)$ about $x = 0$. Write your answer using summation (sigma) notation, including proper indices.

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$$

$$f(x) = 2x \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \right)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{4n+1}}{n!}$$

c. [4 points] Find the Taylor series of $g(x)$ about $x = 0$. Write your answer using summation (sigma) notation, including proper indices.

Solution: Starting with the Taylor series for $e^x$ we have

$$g(x) = \int g'(x) dx$$

$$= \int f(x) dx = \sum_{n=0}^{\infty} (-1)^n \frac{2x^{4n+2}}{(4n+2)n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)n!}$$