6. [8 points] Let $P(t)$ be the population of birds living in a lake t days after January 1, 2009. It has been noticed that the rate of growth of the population of birds varies depending on the season of the year. To take this into consideration, we assume the rate of growth of the population is equal to $k(t)P$, where $k(t) = \frac{1}{100}\sin(\frac{2\pi}{365}t)$. Hence $P(t)$ satisfies

$$\frac{dP}{dt} = k(t)P$$

There were 100 birds living in the lake in January 1, 2009.

a. [6 points] Solve the differential equation satisfied by $P(t)$ and find the population of birds after 100 days after January 1, 2009.

Solution:

$$\frac{dP}{dt} = \frac{1}{100}\sin\left(\frac{2\pi}{365}t\right)P$$

$$\frac{dP}{P} = \frac{1}{100}\sin\left(\frac{2\pi}{365}t\right)dt$$

$$\ln|P(t)| = -\frac{365}{100(2\pi)}\cos\left(\frac{2\pi}{365}t\right) + C$$

$$P(t) = D\ e^{-\frac{365}{100(2\pi)}\cos\left(\frac{2\pi}{365}\ t\right)}$$

Using the initial condition $P(0) = 100$, we get

$$100 = D\ e^{-\frac{365}{100(2\pi)}}$$

$$D = 100\ e^{\frac{365}{100(2\pi)}} = 178.767.$$

$P(t) = 178.767\ e^{-\frac{365}{100(2\pi)}\cos\left(\frac{2\pi}{365}\ t\right)}$ and $P(100) \approx 195$ birds.

b. [2 points] Graph the solution $P(t)$ in your calculator and use this graph to answer the following questions:

1. What is the maximum and minimum amount of birds living in the lake throughout the year?
2. When are the maximum and minimum expected to occur?

Solution: Maximum at $t = \frac{365}{2} = 182.5$ days there are 319 birds.
Minimum at $t = 0$ or $t = 365$ there are 100 birds,