8. [14 points] Consider the following power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n(2n+1)}(x-3)^n$$

a. [11 points] For what values of x does the power series converge?

Solution: Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{4^{n+1}(2n+3)}(x-3)^{n+1}\right|}{\left|\frac{(-1)^n}{4^n(2n+1)}(x-3)^n\right|} = |x-3| \lim_{n\to\infty} \frac{2n+1}{4(2n+3)}$$
$$= |x-3| \lim_{n\to\infty} \frac{2n}{8n} = \frac{|x-3|}{4}$$

Then the series converge for values of x satisfying $\frac{|x-3|}{4} < 1$. Then the series converge for $-1 < x < 7$.

Enpoints ($x = -1, 7$): $x = -1$ yields $\sum_{n=1}^{\infty} \frac{1}{2n+1}$. Limit Comparison Test with $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} == \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2}.$$

Hence $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges.

$x = 7$ yields $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$. Alternating series test: $a_n = f(n) = \frac{1}{2n+1}$

- $\lim_{n\to\infty} \frac{1}{2n+1} = 0$.
- a_n decreasing: $f'(n) = \frac{-2}{(2n+1)^2} < 0$ for $n > 0$.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges. Hence the interval of convergence of the series is $(-1, 7]$.

b. [2 points] For what values of x does the power series converges absolutely?

Solution: $-1 < x < 7$

c. [1 point] For what values of x does the power series converges conditionally?

Solution: $x = 7$