8. [14 points] Consider the following power series

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n(2n+1)}(x-3)^n \]

**a.** [11 points] For what values of \( x \) does the power series converge?

*Solution:* Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\frac{(-1)^{n+1}}{4^{n+1}(2n+3)}(x-3)^{n+1}}{\frac{(-1)^n}{4^n(2n+1)}(x-3)^n} = |x-3| \lim_{n \to \infty} \frac{2n+1}{4(2n+3)}
\]

\[
= |x-3| \lim_{n \to \infty} \frac{2n}{8n} = \frac{|x-3|}{4}
\]

Then the series converge for values of \( x \) satisfying \( \frac{|x-3|}{4} < 1 \). Then the series converge for \(-1 < x < 7\).

Endpoints \((x = -1, 7)\):

- \( x = -1 \) yields \( \sum_{n=1}^{\infty} \frac{1}{2n+1} \). Limit Comparison Test with \( \frac{1}{n} \)

\[
\lim_{n \to \infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{2n+1} = \lim_{n \to \infty} \frac{n}{2n} = \frac{1}{2}
\]

Hence \( \sum_{n=1}^{\infty} \frac{1}{2n+1} \) diverges.

- \( x = 7 \) yields \( \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \). Alternating series test: \( a_n = f(n) = \frac{1}{2n+1} \)

  - \( \lim_{n \to \infty} \frac{1}{2n+1} = 0 \).
  - \( a_n \) decreasing: \( f'(n) = -\frac{2}{(2n+1)^2} < 0 \) for \( n > 0 \).

Hence \( \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \) converges. Hence the interval of convergence of the series is \((-1, 7]\).

**b.** [2 points] For what values of \( x \) does the power series converges absolutely?

*Solution:* \(-1 < x < 7\)

**c.** [1 point] For what values of \( x \) does the power series converges conditionally?

*Solution:* \( x = 7 \)