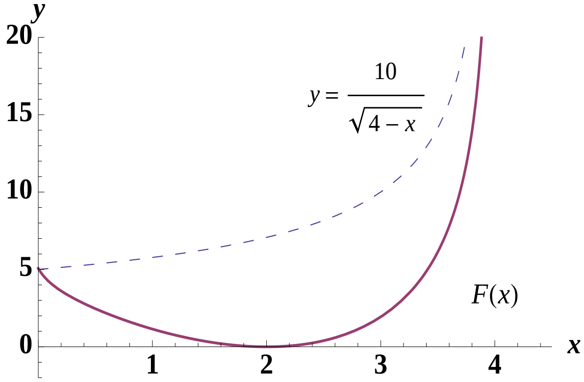
2. [12 points] The graph of $F(x)$ is given below. The function $F(x)$ is defined for $0 \leq x < 4$, and its graph is given below. As shown $F(x)$ has a vertical asymptote at $x = 4$. Let $G(x)$ be the antiderivative of $F(x)$ with $G(1) = 1$.



a. [2 points] For what values of x is $G(x)$ increasing?

Solution: Since $G(x)$ is an antiderivative of $F(x)$, $G'(x) = F(x)$. Then $G(x)$ is increasing for $F(x) \geq 0$, which is true for $0 \leq x < 4$.

b. [2 points] For what values of x is $G(x)$ concave up?

Solution: Since $G'(x) = F(x), G''(x) = F'(x)$. Therefore, $G(x)$ is concave up where $F'(x) \geq 0$, i.e. when $F(x)$ is increasing. This is true for $2 \leq x < 4$.

c. [2 points] Find a formula for $G(x)$ in terms of $F(x)$.

Solution: Since $G(x)$ is the antiderivative of $F(x)$ with $G(1) = 1$, the Construction Theorem (Second Fundamental Theorem of Calculus) gives that $G(x) = 1 + \int_1^x F(t)dt$.

d. [4 points] Is $\int_0^4 \frac{10}{\sqrt{4-x}}dx$ convergent or divergent? If it is convergent, find its exact value.

Solution: The given integral is improper (the integrand is undefined at $x = 4$), so we must use limits to calculate it. Substituting $w = 4 - x, dw = -dx$, we obtain

$$\int_0^4 \frac{10}{\sqrt{4-x}}dx = \lim_{b \to 4^-} \int_0^b \frac{10}{(4-x)^{1/2}}dx = \lim_{b \to 4^-} \int_{w(0)}^{w(b)} \frac{10}{w^{1/2}} \cdot -dw$$

$$= \lim_{b \to 4^-} -20w^{1/2}\Big|_{w(0)}^{w(b)}.$$

In terms of x,

$$\lim_{b \to 4^-} \left(-20\sqrt{4-x}\right)\Big|_0^b = \lim_{b \to 4^-} -20\left(\sqrt{4-b} - \sqrt{4-0}\right) = -20\left(\sqrt{4-4} - \sqrt{4}\right) = 40.$$

So the integral converges.

e. [2 points] Does $\lim_{x \to 4^-} G(x)$ exist? Justify.

Solution: Yes, the given limit exists. This is because $\lim_{x\to 4^-} G(x) = \lim_{x\to 4^-}\left(1+\int_1^x F(t)dt\right) = 1+\int_1^4 F(x)dx$, so the existence of the given limit depends on whether or not the integral $\int_1^4 F(x)dx$ converges. We are given that $F(x) \le \frac{10}{\sqrt{4-x}}$ over $1 \le x < 4$, then

$$\int_1^4 F(x)dx \le \int_1^4 \frac{10}{\sqrt{4-x}}dx$$

and since $\int_1^4 \frac{10}{\sqrt{4-x}}dx$ converges (by a similar calculation to part **d.**), $\int_1^4 F(x)dx$ converges by the comparison method.