5. [10 points] When a voltage V in volts is applied to a series circuit consisting of a resistor with resistance R in ohms and an inductor with inductance L, the current $I(t)$ at time t is given by

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \text{where } V, R, \text{ and } L \text{ are constants.}$$

a. [2 points] Show that $I(t)$ satisfies

$$\frac{dI}{dt} = \frac{V}{L}\left(1 - \frac{R}{V}I\right).$$

Solution:

$$\frac{dI}{dt} = \frac{V}{R}\left(-e^{-Rt/L} \cdot -\frac{R}{L}\right) = \frac{V}{L}e^{-Rt/L} = \frac{V}{L}\left(1 - (1 - e^{-Rt/L})\right)$$
$$= \frac{V}{L}\left(1 - \frac{R}{V}I\right)$$

b. [6 points] Find a Taylor series for $I(t)$ about $t = 0$. Write the first three nonzero terms and a general term of the Taylor series.

Solution: Instead of taking derivatives of $I(t)$ at $t = 0$, we can use the given expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$ about 0. Here, $x = \frac{-Rt}{L}$.

$$\begin{aligned}
I(t) &= \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \\
&= \frac{V}{R}\left[1 - \left(1 + (\frac{-Rt}{L}) + \frac{(-\frac{Rt}{L})^2}{2!} + \frac{(-\frac{Rt}{L})^3}{3!} + \ldots + \frac{(-\frac{Rt}{L})^n}{n!} + \ldots\right)\right] \\
&= \frac{V}{R}\left[1 - \left(1 - \frac{R}{L}t + \frac{R^2}{2!L^2}t^2 - \frac{R^3}{3!L^3}t^3 + \ldots + \frac{(-1)^n R^n}{n!L^n}t^n + \ldots\right)\right] \\
&= \frac{V}{R}\left(\frac{R}{L}t - \frac{R^2}{2!L^2}t^2 + \frac{R^3}{3!L^3}t^3 - \ldots + \frac{(-1)^{n+1}R^n}{n!L^n}t^n + \ldots\right) \\
&= \frac{V}{L}t - \frac{VR}{2!L^2}t^2 + \frac{VR^2}{3!L^3}t^3 - \ldots + \frac{(-1)^{n+1}VR^{n-1}}{n!L^n}t^n + \ldots
\end{aligned}$$

c. [2 points] Use the Taylor series to compute

$$\lim_{t\to 0} \frac{I(t)}{t}.$$

Solution:

$$\begin{aligned}\lim_{t\to 0} \frac{I(t)}{t} &= \lim_{t\to 0} \frac{1}{t}\left(\frac{V}{L}t - \frac{VR}{2!L^2}t^2 + \frac{VR^2}{3!L^3}t^3 - \ldots + \frac{(-1)^{n+1}VR^{n-1}}{n!L^n}t^n + \ldots\right)\\ &= \lim_{t\to 0}\left(\frac{V}{L} - \frac{VR}{2!L^2}t^1 + \frac{VR^2}{3!L^3}t^2 - \ldots + \frac{(-1)^{n+1}VR^{n-1}}{n!L^n}t^{n-1} + \ldots\right)\\ &= \frac{V}{L},\end{aligned}$$

since all of the terms involving t evaluate to 0.