7. [12 points] Determine whether the following series converge or diverge (circle your answer). Be sure to mention which tests you used to justify your answer. If you use the comparison test or limit comparison test, write an appropriate comparison function.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$

Solution: Since

$$\lim_{n\to\infty}\frac{\sqrt{n}}{1+2\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}}{2\sqrt{n}}=\frac{1}{2},$$

then the sequence $a_n=(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$ does not converge to zero (it oscillates closer to $\frac{1}{2}$ and $-\frac{1}{2}$. Since the terms a_n does not converge to 0, then the series $\sum_{n=1}^{\infty}a_n$ diverges.

b. [4 points] $\displaystyle\sum_{n=1}^{\infty}ne^{-n^2}$

Solution: Let $f(x)=xe^{-x^2}$. The function $f(x)>0$ and $f'(x)=e^{-x^2}(1-2x^2)<0$ for $x\geq 1$. Hence by the Integral test

$$\sum_{n=1}^{\infty}ne^{-n^2}\text{ behaves as }\int_1^{\infty}xe^{-x^2}dx=\lim_{b\to\infty}\int_1^b xe^{-x^2}dx=\lim_{b\to\infty}-\frac{1}{2}e^{-x^2}|_1^b=\frac{1}{2e}$$

the series converges.

c. [5 points] $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$

Solution: This series has positive and negative terms. Since

$$\left|\frac{\cos(n^2)}{n^2}\right|\leq\frac{1}{n^2},$$

then the series of the absolute values satisfies

$$\sum_{n=1}^{\infty}\left|\frac{\cos(n^2)}{n^2}\right|\leq\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

The series on the right converges by the p series test with $p=2$, hence the series of absolute values converges. Since the series of absolute values converges, then $\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$ converges.