8. [12 points] Let
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n}$$

a. [3 points] At $x=-3$, does the series converge absolutely, conditionally or diverge?

Solution: At $x=-3$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(-2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$
The sequence $a_n = \frac{1}{n+1}$ is decreasing and converges to 0. By the Alternating series test, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges. The convergence is not absolute since
$$\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{n+1}\right| = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}.$$
which diverges by p series test with $p=1$. Hence the series converges conditionally at $x=-3$.

b. [2 points] Using just your answer in (a), state the possible values for the radius of convergence R could be. Justify.

Solution:
Solution 1: Since the center of the series is $a=-1$ and the series converges at $x=-3$, then $R \geq 2$.
Solution 2: Since power series converges absolutely inside its interval of convergence and at $x=-3$, the series converges conditionally, then $R=2$.

c. [7 points] Find the interval of convergence of the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n}$$

Solution:
Solution 1: Using Ratio Test
$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{4^{n+1}(n+2)}(x+1)^{2n+2}\right|}{\left|\frac{(-1)^n}{4^n(n+1)}(x+1)^{2n}\right|} = \frac{|x+1|^2}{4} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{|x+1|^2}{4} = L.$$
Then the series converges if $\frac{|x+1|^2}{4} < 1$ or if $-3 < x < 1$. We already know from (a) that at $x=-3$ the series converges. If $x=1$, then the series become $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by (a). Hence the interval of convergence is $[-3,-1]$.

Solution 2: Since the radius of convergence is equal to 2, then we only need to check the other endpoint of the interval of convergence $x=1$. If $x=1$, then the series become $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by (a). Hence the interval of convergence is $[-3,-1]$.