4. [11 points] A tank initially has 27 m^3 of water. At $t = 0$ (t in minutes), a pump takes water out of the tank. Let $V(t)$ be the volume of water (in m^3) in the tank t minutes after the pump was activated. Suppose the function $V(t)$ satisfies the differential equation

$$\frac{dV}{dt} = kV^{\frac{1}{3}}$$

where k is a constant.

a. [2 points] Is k positive or negative? What are the units of k?

Solution: $k < 0$, units are $\frac{\text{m}^2}{\text{min}}$

b. [7 points] Find a formula for $V(t)$. Your formula must contain only the constant k and the variable t.

Solution:

$$\begin{aligned}
\frac{dV}{dt} &= kV^{\frac{1}{3}} \\
\int V^{-\frac{1}{3}} dV &= \int k dt \\
\frac{3}{2} V^{\frac{2}{3}} &= kt + C \\
V &= \left(\frac{2}{3}kt + C\right)^{\frac{3}{2}} \\
27 = V(0) = C^{\frac{3}{2}} &\Rightarrow C = 27^{\frac{2}{3}} = 9 \\
V(t) &= \left(\frac{2}{3}kt + 9\right)^{\frac{3}{2}}
\end{aligned}$$

c. [2 points] How long does it take for the tank to empty? Your answer may contain the constant k.

Solution:

$$\begin{aligned}
0 = V(t) &= \left(\frac{2}{3}kt + 9\right)^{\frac{3}{2}} \\
0 &= \frac{2}{3}kt + 9 \\
t &= -\frac{27}{2k}
\end{aligned}$$