9. [11 points] An object is dropped from a height of 100 meters. If air resistance is considered, the height of the object $y(t)$ (in meters) above the ground t seconds after it was dropped is given by

$$y(t) = 100 - \frac{g}{k}t + \frac{g}{k^2}\left(1 - e^{-kt}\right).$$

where $k > 0$ is a constant representing the intensity of air resistance and $g = 9.8$ m/s^2 is the acceleration due to gravity.

a. [3 points] Show that $y(t)$ satisfies $y'' + ky' + g = 0$.

Solution:

$$\begin{aligned} y' &= -\frac{g}{k} + \frac{g}{k^2}ke^{-kt} = -\frac{g}{k}\left(1 - e^{-kt}\right) \\ y'' &= -\frac{g}{k}(ke^{-kt}) = -ge^{-kt} \\ y'' + ky' + g &= -ge^{-kt} - g(1 - e^{-kt}) + g = 0. \end{aligned}$$

b. [6 points] Use the first four nonzero terms of the Taylor series of the function $f(t) = e^{-kt}$ about $t = 0$ to find an approximation for $y(t)$.

Solution:

$$\begin{aligned} e^{-kt} &= 1 - kt + \frac{k^2}{2}t^2 - \frac{k^3}{6}t^3 + \cdots \\ y(t) &= 100 - \frac{g}{k}t + \frac{g}{k^2}\left(1 - 1 + kt - \frac{k^2}{2}t^2 + \frac{k^3}{6}t^3 - \cdots\right) \\ &= 100 - \frac{g}{2}t^2 + \frac{gk}{6}t^3 - \cdots \end{aligned}$$

c. [2 points] Using your answer from part (b), evaluate $\lim_{k \to 0} y(t)$.

Solution:

$$\lim_{k \to 0} y(t) = \lim_{k \to 0} 100 - \frac{g}{2}t^2 + \frac{gk}{6}t^3 - \cdots = 100 - \frac{g}{2}t^2.$$