2. [11 points] Determine the convergence or divergence of the following series. In parts (a) and (b), support your answers by stating and properly justifying any test(s), facts or computations you use to prove convergence or divergence. Circle your final answer. Show all your work.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ CONVERGES **DIVERGES**

Solution:

$$\lim_{n\to\infty} \frac{9n}{e^{-n}+n} = \lim_{n\to\infty} \frac{9n}{n} = 9 \neq 0.$$

Since $\lim_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ diverges.

b. [4 points] $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$. **CONVERGES** DIVERGES

Solution: The function $f(n) = \frac{4}{n(\ln n)^2}$ is positive and decreasing for $n > 2$, then by Integral Test the convergence or divergence of $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$ can be determined with the convergence or divergence of $\displaystyle\int_2^{\infty} \frac{4}{x(\ln x)^2} dx$

$$\int \frac{4}{x(\ln x)^2} dx = \int \frac{4}{u^2} du \qquad \text{where } u = \ln x.$$

$$= -\frac{4}{u} + C = -\frac{4}{\ln x} + C$$

Hence

$$\int_2^{\infty} \frac{4}{x(\ln x)^2} dx = \lim_{b\to\infty} -\frac{4}{\ln x}\Big|_2^b = -\frac{4}{\ln 2} \qquad \text{converges.}$$

or

$$\int_2^{\infty} \frac{4}{x(\ln x)^2} dx = 4\int_{\ln 2}^{\infty} \frac{1}{u^2} du \qquad \text{converges by } p\text{-test with } p = 2 > 1.$$

c. [4 points] Let r be a **real** number. For which values of r is the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^r+4}$ absolutely convergent? Conditionally convergent? No justification is required.

Solution:

Absolutely convergent if : $r > 3$

Conditionally convergent if : $2 < r \leq 3$

Justification (not required):

- Absolute convergence:
  The series $\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^2}{n^r+4}\right| = \sum_{n=1}^{\infty}\frac{n^2}{n^r+4}$ behaves like $\sum_{n=1}^{\infty}\frac{n^2}{n^r} = \sum_{n=1}^{\infty}\frac{1}{n^{r-2}}$. The last series is a p-series with $p = r - 2$ which converges if $r - 2 > 1$. Hence the series converges absolutely if $r > 3$.

- Conditionally convergence:
  The function $\frac{n^2}{n^r+4}$ is positive and decreasing (for large values of n) when $r > 2$. Hence by the Alternating series test $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^r+4}$ converges in this case.