8. [12 points]

a. [4 points] Let a be a positive constant. Determine the first three nonzero terms of the Taylor series for

$$f(x) = \frac{1}{(1+ax^2)^4}$$

centered at $x = 0$. Show all your work. Your answer may contain a.

Solution: Using the binomial series $(1+u)^p$ with $u = ax^2$ and $p = -4$

$$f(x) = \frac{1}{(1+ax^2)^4} \approx 1 + pu + \frac{p(p-1)}{2}u^2 = 1 - 4ax^2 + 10a^2x^4$$

b. [2 points] What is the radius of convergence of the Taylor series for $f(x)$? Your answer may contain a.

Solution: Since the interval of convergence of the binomial series is $-1 < u < 1$, then the interval of convergence of the series for $f(x)$ is $-1 < ax^2 < 1$, or $-\sqrt{\frac{1}{a}} < x < \sqrt{\frac{1}{a}}$. Hence the radius of convergence is $R = \sqrt{\frac{1}{a}}$.

c. [3 points] Determine the first three nonzero terms of the Taylor series for

$$g(t) = \int_0^t \frac{1}{(1+ax^2)^4} dx,$$

centered at $t = 0$. Show all your work. Your answer may contain a.

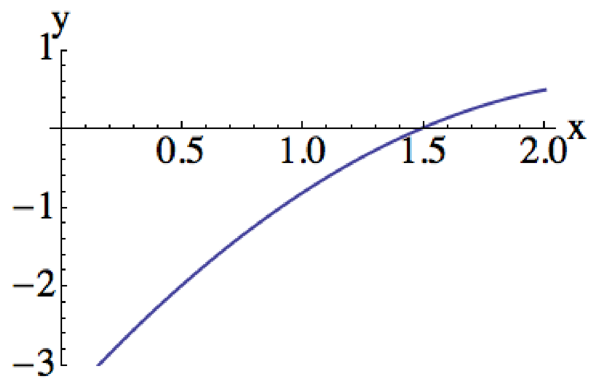
Solution:

$$g(t) = \int_0^t \frac{1}{(1+ax^2)^4} dx \approx \int_0^t 1 - 4ax^2 + 10a^2x^4 dx = x - \frac{4}{3}ax^3 + 2a^2x^5|_0^t = t - \frac{4}{3}at^3 + 2a^2t^5$$

d. [3 points] The degree-2 Taylor polynomial of the function $h(x)$, centered at $x = 1$, is

$$P_2(x) = a + b(x-1) + c(x-1)^2.$$

The following is a graph of $h(x)$:



What can you say about the values of a, b, c? You may assume a, b, c are nonzero. Circle your answers. No justification is needed.

Solution:

a is:	Positive	**NEGATIVE**	Not enough information
b is:	**POSITIVE**	Negative	Not enough information
c is:	Positive	**NEGATIVE**	Not enough information