12. [8 points] Franklin, your friendly new neighbor, is building a large chicken sanctuary. You decide to help Franklin build a special chicken coop with volume (in cubic km) given by the integral

$$\int_0^1 x\sqrt{1-\cos(x^2)}\,dx.$$

This integral is difficult to evaluate precisely, so you decide to use the methods you've learned this semester to help out Franklin. Your friend and president-elect, Kazilla, stops by to give you a hand. She suggests finding the 4th degree Taylor polynomial, $P_4(x)$, for the function $1-\cos(x^2)$ near $x=0$.

a. [4 points] Find $P_4(x)$.

Solution: We can use the Taylor series expansion for $\cos(x^2)$

$$\cos(y) = \sum_{n=0}^{\infty} \frac{(-1)^n\, y^{2n}}{(2n)!} = 1 - \frac{y^2}{2!} + \cdots + \frac{(-1)^n\, y^{2n}}{(2n)!} + \cdots$$

so

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n\, (x^2)^{2n}}{(2n)!} = 1 - \frac{(x^2)^2}{2!} + \cdots + \frac{(-1)^n\, (x^2)^{2n}}{(2n)!} + \cdots$$

Therefore

$$P_4 = 1 - (1 - \frac{x^4}{2!}) = \frac{x^4}{2!}$$

b. [4 points] Substitute $P_4(x)$ for $1-\cos(x^2)$ in the integral and compute the resulting integral by hand, showing all of your work.

Solution:

$$\begin{aligned}\int_0^1 x\sqrt{P_4(x)}dx &= \int_0^1 x\sqrt{\frac{x^4}{2}} \\ &= \int_0^1 \frac{x^3}{\sqrt{2}} \\ &= \left.\frac{x^4}{4\sqrt{2}}\right|_{x=0}^{1} \\ &= \frac{1}{4\sqrt{2}}\end{aligned}$$