12. [8 points] Franklin, your friendly new neighbor, is building a large chicken sanctuary. You decide to help Franklin build a special chicken coop with volume (in cubic km) given by the integral

\[ \int_0^1 x \sqrt{1 - \cos(x^2)} \, dx. \]

This integral is difficult to evaluate precisely, so you decide to use the methods you’ve learned this semester to help out Franklin. Your friend and president-elect, Kazilla, stops by to give you a hand. She suggests finding the 4th degree Taylor polynomial, \( P_4(x) \), for the function \( 1 - \cos(x^2) \) near \( x = 0 \).

a. [4 points] Find \( P_4(x) \).

**Solution:** We can use the Taylor series expansion for \( \cos(x^2) \)

\[
\cos(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} = 1 - \frac{y^2}{2!} + \cdots + \frac{(-1)^n y^{2n}}{(2n)!} + \cdots
\]

so

\[
\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = 1 - \frac{(x^2)^2}{2!} + \cdots + \frac{(-1)^n (x^2)^{2n}}{(2n)!} + \cdots
\]

Therefore

\[
P_4 = 1 - \left(1 - \frac{x^4}{2!}\right) = \frac{x^4}{2!}
\]

b. [4 points] Substitute \( P_4(x) \) for \( 1 - \cos(x^2) \) in the integral and compute the resulting integral by hand, showing all of your work.

**Solution:**

\[
\int_0^1 x \sqrt{P_4(x)} \, dx = \int_0^1 x \sqrt{\frac{x^4}{2}}
\]

\[
= \int_0^1 \frac{x^3}{\sqrt{2}}
\]

\[
= \left. \frac{x^4}{4\sqrt{2}} \right|_0^1
\]

\[
= \frac{1}{4\sqrt{2}}
\]