8. [7 points] Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

a. [2 points] At which x-value is the interval of convergence of this power series centered?

Solution: This power series is centered on $x = -2$.

b. [5 points] The radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$ is 3. Find the interval of convergence for this power series. Thoroughly justify your answer.

Solution: Since the radius of convergence for this power series is 3 and it is centered on $x = 2$, the interval of convergence contains the open interval $(-2-3, -2+3) = (-5, 1)$. Now we only need to check the endpoints $x = -5$ and $x = 1$.

- For $x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(1+2)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test with $p = 1$ (this is the harmonic series).
- For $x = -5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-5+2)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test.

Therefore, the interval of convergence for this power series is $[-5, 1)$.

9. [5 points] Find the radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^{2n}$

Solution: Let the n-th term be denoted by a_n

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \left| \frac{(2(n+1))! x^{2(n+1)}}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)! x^{2n}} \right| \\ &= \left| \frac{(2n+2)(2n+1)x^2}{((n+1)^2} \right|\end{aligned}$$

Therefore, we can use the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)x^2}{((n+1)^2} \right| = 4x^2.$$

So this series converges for x with $4x^2 < 1$, or rather with $x^2 < \frac{1}{4}$ which implies that the radius of convergence is $1/2$.