8. [9 points] Consider the function $g(x)$ defined by the power series

$$g(x) = \sum_{n=0}^{\infty} \frac{2^n (n!)^2 x^n}{(2n)!}.$$

a. [6 points] Find the **radius** of convergence of the power series. You do not need to find the interval of convergence.

Solution: Applying the ratio test

$$\lim_{n\to\infty} \left| \frac{\left(\frac{2^{n+1}((n+1)!)^2 x^{n+1}}{(2(n+1))!}\right)}{\left(\frac{2^n (n!)^2 x^n}{(2n)!}\right)} \right| = \lim_{n\to\infty} \frac{2(n+1)^2|x|}{(2n+2)(2n+1)}$$
$$= \frac{|x|}{2}.$$

Therefore the series converges whenever $\frac{|x|}{2} < 1$, or $|x| < 2$. Hence the radius of convergence is 2.

b. [3 points] Use the first 3 nonzero terms of the power series to estimate

$$\int_0^1 \frac{g(x) - 1}{x}\, dx.$$

Solution: Since $g(x) = 1 + \frac{2}{2!}x + \frac{2^2(2!)^2}{4!}x^2 + \cdots = 1 + x + \frac{2}{3}x^2 + \ldots$,

$$\int_0^1 \frac{g(x)-1}{x}\, dx \approx \int_0^1 \frac{1 + x + \frac{2}{3}x^2 - 1}{x}\, dx = \int_0^1 (1 + \frac{2}{3}x)\, dx = \frac{4}{3}.$$