10. [12 points]

a. [6 points] Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n \cdot 3^n}.$$

Show every step of any calculations and fully justify your answer with careful reasoning. Write your final answers on the answer blanks provided.

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{2(n+1)}}{(n+1)3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^{n+1}x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(-1)^{n+1}} \cdot \frac{x^{2n+2}}{x^{2n}} \cdot \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}}\right| = \left|\frac{-x^2}{3}\right| \lim_{n\to\infty} \cancel{\frac{n}{n+1}} \, 1$$

$= x^2/3$, which is <1 when $|x|<\sqrt{3}$.

$x = \pm\sqrt{3} \Rightarrow$ sum $= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which converges by alternating series test

AST:
alternating ✓
|terms| decreasing ✓
terms → 0 ✓

Answer: Radius of Convergence: $\boxed{\sqrt{3}}$

Interval of Convergence: $\boxed{[-\sqrt{3}, \sqrt{3}]}$

b. [3 points] The Maclaurin series for a function $f(x)$ is the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n \cdot 3^n}$.

The function $f(x)$ is closely related to one of the functions that appears on the formula sheet on the last page of this exam. Find a formula for $f(x)$ in closed form (i.e. without sigma notation or ellipses $(\cdots)$).

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n \cdot 3^n} = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{x^2}{3}\right)^n = \ln\left(1+\frac{x^2}{3}\right)$$

Answer: $f(x) = \boxed{\ln(1+x^2/3)}$

c. [3 points] Suppose the Taylor series about $x = 0$ for a function $g(x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^{3n}$.

Let $h(x) = g(\frac{x}{2})$. Find $h^{(15)}(0)$.

$$h(x) = \sum \frac{(-1)^n}{n^2}\left(\frac{x}{2}\right)^{3n} = \sum \frac{(-1)^n}{n^2(2^{3n})} x^{3n}.$$

$\frac{h^{(15)}(0)}{15!}$ = coefficient of x^{15} $= \frac{(-1)^5}{5^2(2^{15})}$

Answer: $h^{(15)}(0) = \boxed{-\frac{15!}{5^2 2^{15}}}$