7. [9 points] The "Witch of Maria Agnesi" is the family of curves described by the parametric equations

$$x = 2at \qquad \text{and} \qquad y = \frac{2a}{1+t^2} = 2a(1+t^2)^{-1}$$

for all t, where a is a positive constant.

a. [4 points] Consider a "Witch curve" as defined above.

i. Find a formula for $\frac{dy}{dx}$ in terms of t and/or a.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\cancel{2a}(1+t^2)^{-2}(2t)}{\cancel{2a}}$$

Answer: $\frac{dy}{dx}$ $\boxed{\frac{-2t}{(1+t^2)^2}}$

ii. Find a formula for the tangent line to the "Witch curve" at the point where $t = 1$. Your answer might involve the constant a but should **not** involve t.

at $t=1$:

$x = 2a(1) = 2a$

$y = \frac{2a}{1+(1)^2} = a$

$$\frac{dy}{dx} = \frac{-2(1)}{(1+1^2)^2} = -\frac{1}{2}$$

Answer: $y =$ $\boxed{a - \frac{1}{2}(x-2a)}$

b. [5 points] The total area in the first quadrant that is bounded between a "Witch curve" and the x-axis is represented by the improper integral $\displaystyle\int_0^\infty \frac{8a^3}{x^2+4a^2}\,dx$.

Determine whether this improper integral converges or diverges.

- If the integral converges, circle "converges", find its exact value (i.e. no decimal approximations), and write the exact value on the answer blank provided. (The exact value may involve the constant a.)
- If the integral diverges, circle "diverges" and carefully justify your answer.

In either case, **you must show all your work carefully using correct notation**. Any direct evaluation of integrals must be done **without using a calculator.**

Converges to $\boxed{2\pi a^2}$ **Diverges**

Hint: Note that $\frac{d}{dx}\left(4a^2 \arctan\left(\frac{x}{2a}\right)\right) = \frac{8a^3}{x^2+4a^2}$.

$$\int_0^\infty \frac{8a^3}{x^2+4a^2}\,dx = \lim_{b\to\infty}\left[4a^2\arctan\left(\frac{x}{2a}\right)\right]_0^b$$

$$= 4a^2 \lim_{b\to\infty} \arctan\left(\frac{b}{2a}\right) - \cancel{\arctan(0)}^{0} = 4a^2\left(\frac{\pi}{2}\right)$$