8. [11 points] Imagine that a one pound ball is attached to a spring. This ball is allowed to move forward and backward on a table, but not up and down (or side to side). When the spring is not stretched at all, we say that the ball is at its starting position. Let $x$ be the displacement of the ball from its starting position in the forward/backward direction. (The value of $x$ is positive if the ball has moved forward from its starting position and negative if the ball has moved backward from its starting position.)

a. [4 points] Let $F(x)$ be the magnitude of the force, measured in pounds, that the spring exerts on the ball when the ball has been pulled $x$ feet from its starting position. Suppose $F(x) = 5x$.

i. Which of the following best estimates the work, in foot-pounds, needed to move the ball a very small distance $\Delta x$ feet forward from a position $x$? Circle ONE choice.

   I. $5$ II. $5x$ III. $2.5x^2$ IV. $5\Delta x$ V. $5x\Delta x$ VI. $2.5x^2\Delta x$

ii. Use your answer to part i. to write an expression involving one or more integrals that gives the total work needed to move the ball from its starting position forward a distance of one half of one foot (i.e. 6 inches). Then compute the value of your integral (either by hand or using your calculator). Include units on your answer.

   Answer: Integral Expression: 

   Numerical Answer (with units): 

b. [4 points] After stretching the spring as described above, you release it from a starting position of $x = 1/2$. The ball oscillates backwards and forwards (in the $x$-direction), and its position $x = x(t)$ satisfies the differential equation $x'' + 5x = 0$. Note that $x'' = \frac{d^2x}{dt^2}$.

For what values of $A$, $B$, and $k$ will the function

$$x(t) = A \sin(kt) + B \cos(kt)$$

be a solution to the differential equation $x'' + 5x = 0$ with the initial conditions $x(0) = 1/2$ and $x'(0) = 0$?

$$x'(t) = kA\cos(kt) - kB\sin(kt) \quad \frac{1}{2} = x(0) = B$$
$$x''(t) = -k^2A\sin(kt) - k^2B\cos(kt) \quad 0 = x'(0) = kA$$

Answer: $A = \boxed{0}$ and $B = \boxed{\frac{1}{2}}$ and $k = \boxed{\pm \sqrt{5}}$

c. [3 points] Using the particular solution that you found in part b, find the first time $t > 0$ when the ball reaches the position $x = 0$.

$$x(t) = \frac{1}{2} \cos(\sqrt{5}t) = 0 \Rightarrow \sqrt{5}t = \frac{\pi}{2} + m\pi$$

Answer: $t = \boxed{\frac{\pi}{2\sqrt{5}}}$