8. [11 points] Imagine that a one pound ball is attached to a spring. This ball is allowed to move forward and backward on a table, but not up and down (or side to side). When the spring is not stretched at all, we say that the ball is at its *starting position*. Let x be the displacement of the ball from its starting position in the forward/backward direction. (The value of x is positive if the ball has moved forward from its starting position and negative if the ball has moved backward from its starting position.)

a. [4 points] Let $F(x)$ be the magnitude of the force, measured in pounds, that the spring exerts on the ball when the ball has been pulled x feet from its starting position. Suppose $F(x) = 5x$.

i. Which of the following best estimates the work, in foot-pounds, needed to move the ball a very small distance Δx feet forward from a position x? *Circle* ONE *choice.*

I. 5 II. $5x$ III. $2.5x^2$ IV. $5\Delta x$ $\boxed{\text{V. } 5x\Delta x}$ VI. $2.5x^2\Delta x$

ii. Use your answer to part i. to write an expression involving one or more integrals that gives the total work needed to move the ball from its starting position forward a distance of one half of one foot (i.e. 6 inches). Then compute the value of your integral (either by hand or using your calculator). Include units on your answer.

$$\int_0^{.5} 5x\,dx = \frac{5}{2}x^2\Big|_0^{.5} = \frac{5}{2}\left(\frac{1}{2}\right)^2$$

Answer: Integral Expression: $\boxed{\int_0^{.5} 5x\,dx}$

Numerical Answer (with units): $\boxed{5/8 \text{ Foot pounds}}$

b. [4 points] After stretching the spring as described above, you release it from a starting position of $x = 1/2$. The ball oscillates backwards and forwards (in the x-direction), and its position $x = x(t)$ satisfies the differential equation $x'' + 5x = 0$. Note that $x'' = \frac{d^2x}{dt^2}$. For what values of A, B, and k will the function

$$x(t) = A\sin(kt) + B\cos(kt)$$

be a solution to the differential equation $x'' + 5x = 0$ with the initial conditions $x(0) = 1/2$ and $x'(0) = 0$?

Since $x'' = -k^2 x$, $k^2 = 5$

$x'(t) = kA\cos(kt) - kB\sin(kt)$ $\frac{1}{2} = x(0) = B$

$x''(t) = -k^2 A\sin(kt) - k^2 B\cos(kt)$ $0 = x'(0) = kA$

Answer: $A = \boxed{0}$ and $B = \boxed{1/2}$ and $k = \boxed{\pm\sqrt{5}}$

c. [3 points] Using the particular solution that you found in part **b**, find the first time $t > 0$ when the ball reaches the position $x = 0$.

$x(t) = \frac{1}{2}\cos(\sqrt{5}t) = 0 \Rightarrow \sqrt{5}t = \frac{\pi}{2} + m\pi$

Answer: $t = \boxed{\pi/2\sqrt{5}}$