10. [6 points] The Taylor series centered at $x = 0$ for a function $F(x)$ converges to $F(x)$ for $-e^{-1} < x < e^{-1}$ and is given below.

$$F(x) = \sum_{n=0}^{\infty} \frac{(n+1)^n}{n!} x^n \quad \text{for } -\frac{1}{e} < x < \frac{1}{e}.$$

a. [2 points] What is $F^{(2018)}(0)$? Make sure your answer is exact. You do not need to simplify.

$$\frac{F^{(2018)}(0)}{2018!} = \text{coefficient of } x^{2018} = \frac{(2018+1)^{2018}}{(2018)!}$$

Answer: $F^{(2018)}(0) = 2019^{2018}$

b. [4 points] Use appropriate Taylor series for $F(x)$ and $\cos(x)$ to compute the following limit:

$$\lim_{x\to 0} \frac{(F(x)-1)(\cos(x)-1)}{x^3}$$

Show your work carefully.

$$F(x) = \frac{(0+1)^0}{0!} x^0 + \frac{(1+1)^1}{1!} x^1 + \frac{(2+1)^2}{2!} x^2 + \cdots$$

$$= 1 + 2x + \frac{9}{2}x^2 + \cdots$$

So $F(x) - 1 = 2x + \frac{9}{2}x^2 + \cdots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

so $\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \cdots$

so we have $\lim_{x\to 0} \frac{(2x + \frac{9}{2}x^2 + \cdots)(-\frac{x^2}{2} + \frac{x^4}{24} - \cdots)}{x^3} = \lim_{x\to 0} \frac{-x^3 + (\text{powers of } x \text{ bigger than } 3)}{x^3}$

Answer: $\lim_{x\to 0} \frac{(F(x)-1)(\cos(x)-1)}{x^3} = -1$