5. [10 points]

a. [5 points] Using an appropriate Taylor series for e^x, determine whether the integral $\displaystyle\int_0^1 \frac{1}{e^{\sqrt{x}}-1}\,dx$ converges.

Circle one: $\boxed{\text{CONVERGES}}$ DIVERGES

Justification:

$$e^{\sqrt{x}} = 1 + \sqrt{x} + \frac{1}{2!}(\sqrt{x})^2 + \frac{1}{3!}(\sqrt{x})^3 + \cdots \geq 1+\sqrt{x} \quad \text{if } 0<x\leq 1.$$

So $e^{\sqrt{x}}-1 \geq (1+\sqrt{x})-1 = \sqrt{x}$

So $\dfrac{1}{e^{\sqrt{x}}-1} \leq \dfrac{1}{\sqrt{x}}$

$\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 \frac{1}{x^{1/2}}dx$ converges by the p-test ($p=1/2$)

So $\displaystyle\int_0^1 \frac{1}{e^{\sqrt{x}}-1}dx$ converges by comparison.

b. [5 points] Using a Taylor series for a function $f(x)$, compute the exact value of $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^{n-1}}$.

(Hint: Consider the function $\dfrac{1}{1-x}$.)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Take derivatives of both sides:

$$-(1-x)^{-2}(-1) = \sum_{n=0}^{\infty} n x^{n-1}$$

. The $n=0$ term is 0 so can ignore it:

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

That converges by the ratio test when $|x|<1$. So if $x=\frac{1}{3}$,

$$\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^{n-1} = \frac{1}{(1-\frac{1}{3})^2}$$

Answer: This is a value of a Taylor series for the function $f(x) =$ $\dfrac{1}{(1-x)^2}$

The radius of convergence of this Taylor series is 1 (No justification needed)

Finally, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^{n-1}} =$ $9/4 = 2.25$