9. [12 points] For each of the questions on this page:
You must circle at least one choice to receive any credit.
No credit will be awarded for unclear markings. No justification is necessary.

For parts **a** - **c** below, circle all of the available correct answers, and circle "NONE OF THESE" if none of the available options are correct.

a. [4 points] Suppose a_n and b_n are nonzero sequences. Functions P and Q satisfy the following: $P(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ for $-1 < x \leq 3$ and $Q(x) = \sum_{n=0}^{\infty} b_n x^n$ for $-1 \leq x \leq 1$. Which of the following must be true?

i. The radius of convergence of the Taylor series for $P(x)$ around $x = 1$ is at least 1.

$\left|\frac{b_n}{n}\right| < |b_n|$

ii. $\sum_{n=1}^{\infty} \frac{b_n}{n}$ converges.

iii. $\sum_{n=0}^{\infty} a_n 2^n$ diverges.

iv. $\sum_{n=0}^{\infty} \frac{1}{a_n}$ diverges.

Since $P(2) = \sum a_n$ converges, $a_n \to 0$ so $\frac{1}{a_n} \to \infty$

v. The Taylor series for $P(x)$ around $x = 0$ is $\sum_{n=0}^{\infty} a_n x^n$.

vi. NONE OF THESE

b. [4 points] Suppose $f(x)$ is a positive, decreasing, and concave up function. Suppose further that all derivatives of $f(x)$ exist at $x = 0$. Define $F(x) = \int_0^x f(t)\,dt$.
Which of the following must be true?

f pos, dec $\Rightarrow$ F inc, conc Down

f dec, conc up $\Rightarrow$ f' neg, inc

False $\rightarrow$ i. TRAP(n) is an overestimate of $\int_0^1 F(x)\,dx$ for all positive integers n.

Since F Conc Dn

if $F = \sum a_n x^n$, $f = \sum n a_n x^{n-1}$

Ratio test: $\sum a_n x^n$ conv if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| < 1$

$\sum n a_n x^{n-1}$ conv if $\lim_{n\to\infty} \left|\frac{(n+1)a_{n+1}}{n a_n}\right| |x| < 1$,

but those limits are the same.

F, f' both inc

ii. $F(x) + F''(x)$ is an increasing function.

iii. The Taylor series for $F(x)$ and for $f(x)$ centered around $x = 0$ both have the same radius of convergence.

$\int_0^1 \frac{f(x)}{F(x)}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{f(x)}{F(x)}\,dx$

let $w = F(x) \Rightarrow dw = f(x)\,dx$

$= \lim_{a\to 0^+} \int_{F(a)}^{F(1)} \frac{dw}{w}$

$= \int_0^{F(1)} \frac{dw}{w}$

which diverges by p-test ($p=1$)

iv. $\int_0^1 \frac{f(x)}{F(x)}\,dx$ converges.

v. $\sum_{n=1}^{\infty} f(n)$ converges.

counterexample: $f(x) = 1 + e^{-x}$

vi. NONE OF THESE

c. [4 points] Consider the differential equation $y' = (\cos(x) - \sin(y))^2$, and suppose $y = g(x)$ is the solution to this differential equation that passes through the point $(0, 0)$. Which of the following must be true?

i. This differential equation has no equilibrium solutions.

ii. $g''(0) = -2$.

iii. $y = \arcsin(\cos(x))$ is an equilibrium solution.

iv. $g(x) \leq 4x$ for all $x > 0$

because $y' \leq 4$

v. $g(x)$ is increasing.

because $y' \geq 0$

vi. NONE OF THESE