3. [10 points] The Taylor series centered at 3 for a function $g(x)$ is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^{2n}}{n\,4^n}.$$

a. [5 points] Determine the radius of convergence for this Taylor series. Show all work.

Solution: Let $a_n = \frac{(-1)^n}{n\,4^n}(x-3)^{2n}$. Then we will find the radius of convergence by applying the ratio test to $\sum_{n=1}^{\infty} a_n$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)4^{n+1}}|x-3|^{2n+2}}{\frac{1}{n4^n|x-3|^{2n}}} \\
&= \lim_{n\to\infty} \frac{n}{4(n+1)}|x-3|^2 \\
&= \frac{1}{4}|x-3|^2
\end{aligned}$$

This is less than 1 when $|x-3|^2 < 4$, or, equivalently, when $|x-3| < 2$. This means that $\sum_{n=0}^{\infty} a_n$ converges when $|x-3| < 2$, and the radius is 2.

Radius: 2

b. [2 points] Which of the following best describes the concavity of $g(x)$ at $x = 3$? Circle the <u>one</u> best answer. No justification is necessary.

CONCAVE UP   [CONCAVE DOWN]   NEITHER   CANNOT BE DETERMINED

c. [3 points] Find $g^{(1010)}(3)$.

Solution: Using the formula for Taylor series, we know that the degree 1010 term will be $\frac{g^{(1010)}(3)}{1010!}(x-3)^{1010}$. With the formula we've been given for this problem, we see that the degree 1010 term happens when $n = 1010/2 = 505$, and will be $\frac{-(x-3)^{1010}}{505 \cdot 4^{505}}$. Therefore

$$\begin{aligned}
\frac{g^{(1010)(3)}}{1010!}(x-3)^{1010} &= \frac{-(x-3)^{1010}}{505 \cdot 4^{505}} \\
\frac{g^{(1010)}(3)}{1010!} &= -\frac{1}{505 \cdot 4^{505}} \\
g^{(1010)}(3) &= -\frac{1010!}{505 \cdot 4^{505}}
\end{aligned}$$

$g^{(1010)}(3) =$ $-\dfrac{1010!}{505 \cdot 4^{505}}$