7. [11 points] Some values of a function $m(x)$ and its derivatives are given below.

x	0	2
$m(x)$	4	1
$m'(x)$	-1	0
$m''(x)$	0	0
$m'''(x)$	3	-2
$m''''(x)$	5	8

a. [4 points] Find a formula for $P_4(x)$, the Taylor polynomial of degree 4 for $m(x)$ about $x = 2$.

Answer: $P_4(x) =$ $1 - \frac{2}{3!}(x-2)^3 + \frac{8}{4!}(x-2)^4$

b. [3 points] Use your answer to approximate the value of $\int_1^3 m(x)\,dx$. Show your work.

Solution:

$$\int_1^3 1 - \frac{2}{3!}(x-2)^3 + \frac{8}{4!}(x-2)^4 dx = x - \frac{2}{4!}(x-2)^4 + \frac{8}{5!}(x-2)^5\Big|_1^3$$
$$= (1 - \frac{2}{4!} + \frac{8}{5!}) - (-1 - \frac{2}{4!} - \frac{8}{5!})$$
$$= 2 + \frac{16}{5!} = \frac{32}{15}$$

Answer: $\int_1^3 m(x)\,dx \approx$ $\frac{32}{15}$

c. [4 points] Let $G(x)$ be the antiderivative of the function $g(x) = m(3x^2)$ with $G(0) = 5$. Find the first three nonzero terms of the Taylor series for $G(x)$ about $x = 0$.

Solution:
Using the table, we see that the first 3 nonzero terms of the Taylor series for $m(x)$ about $x = 0$ are $4 - x + \frac{1}{3!}x^3$. Then the first 3 nonzero terms for the Taylor series for $m(3x^2)$ about $x = 0$ are $4 - 3x^2 + \frac{1}{3!}(3x^2)^3$. To get the Taylor series for an antiderivative of $m(3x^2)$ about $x = 0$, we take an antiderivative of the Taylor series we found above: $C + 4x - \frac{3}{3}x^3$. Since $G(0) = 5$, we must have $C = 5$.

Answer: $5 + 4x - x^3$