8. [9 points] Derivative Girl and Gradi-Ant are excited for the end of the semester. To celebrate, they decide to make an Infinite Party Horn. In this problem, \( x \) and \( y \) are measured in meters. (In Derivative Girl’s world, infinite objects are possible.)

a. [4 points] They decide to make the horn by rotating the region bounded by the positive \( x \)-axis, the positive \( y \)-axis, and the function \( y = \frac{1}{2(x+1)^2} \) about the line \( y = -1 \). Write, but do not evaluate, an expression involving one integral that gives the volume, in cubic meters, of the Infinite Party Horn.

Answer:
\[
\pi \int_{0}^{\infty} \left( \frac{1}{2(x+1)^2} + 1 \right)^2 - 1 \, dx
\]

b. [5 points] Derivative Girl will use her favorite continuous and differentiable functions \( f \) and \( g \) to make a banner for the Infinite Party Horn. She loves the functions \( f \) and \( g \) because they have the properties:

- \( \frac{d}{dx} \left( \frac{1+x}{g(x)} \right) = f(x) \),
- \( g(1) = 15 \),
- \( \lim_{x \to \infty} g(x) = \infty \),
- \( \lim_{x \to \infty} g'(x) = 5 \),

and the area of the banner, in square meters, is given by
\[
\int_{1}^{\infty} 20f(x) \, dx.
\]

Does the banner have finite area? If so, what is the banner’s area? Show all work and indicate any theorems you use.

Solution: Using the information we’ve been given, we find
\[
\int_{1}^{\infty} 20f(x) \, dx = \lim_{b \to \infty} \int_{1}^{b} 20f(x) \, dx
\]
\[
= \lim_{b \to \infty} \frac{20 + 20b}{g(b)} - \frac{40}{g(1)}
\]
\[
= \lim_{b \to \infty} \frac{20}{g'(b)} - \frac{40}{15} \text{ by L'Hopital’s Rule}
\]
\[
= \frac{20}{5} - \frac{40}{15}
\]

Answer (Circle one):  Infinite area  Finite area:  \( \frac{4}{7} \) m\(^2\)