3. [19 points] Consider the function $B(x)$ described on its domain by its Taylor series around $x = 0$,

$$B(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!(2n)!} x^{2n}.$$

a. [5 points] Find the first four non-zero terms of the Taylor series for $B(x)$ about $x = 0$. You do not need to evaluate any factorials in your answer.

Solution:

$$B(x) = 1 - \frac{1}{2!2!} x^2 + \frac{1}{3!4!} x^4 - \frac{1}{4!6!} x^6 + \ldots$$

b. [6 points] Find the radius of convergence of the Taylor series. Show all of your work and use proper notation.

Solution: We use the ratio test with $a_n = \frac{(-1)^n}{(n+1)!(2n)!} x^{2n}$.
We have:

$$\begin{aligned} \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+1)!(2n)!|x^{2n+2}|}{(n+2)!(2n+2)!|x^{2n}|} \\ &= \lim_{n\to\infty} \frac{x^2}{(n+2)(2n+2)(2n+1)} \\ &= 0 \text{ for all } x. \end{aligned}$$

Therefore, the radius of convergence is ∞.

c. [3 points] Is $B(x)$ an odd function, an even function, or neither? Explain your reasoning.

Solution: The Taylor series contains only even powers of x, and so $B(x)$ is an even function.

d. [5 points] Find the value of $B^{(2020)}(0)$. You do not need to evaluate any factorials in your answer.

Solution: We know that $\frac{B^{(2020)}(0)}{2020!}x^{2020}$ appears in the Taylor series for $B(x)$. Comparing this with the given expression for $B(x)$, we see we must have $x^{2n} = x^{2020}$, i.e. $n = 1010$. Therefore, comparing coefficients of x^{2020},

$$\frac{B^{(2020)}(0)}{2020!} = \frac{(-1)^{1010}}{1011!2020!},$$

and so

$$B^{(2020)}(0) = \frac{1}{1011!}.$$