6. [11 points]

a. [6 points] Find the Taylor series about $x = 0$ for the function $f(x) = 3 + \cos(2x^2)$. Write your answer using sigma notation and also write out the first **three** non-zero terms. You do not need to simplify any factorials or exponentials that appear in your answer.

Solution: We have $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, and so,

$$\cos(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!}.$$

So

$$3 + \cos(2x^2) = 3 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!} = 4 - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!} - \ldots$$

$$= 4 - \frac{2^2 x^4}{2!} + \frac{2^4 x^8}{4!} - \ldots$$

b. [5 points] The function $f(x)$ from part a) has an antiderivative $F(x)$ which satisfies $F(0) = 9$. Find the first four nonzero terms of the Taylor series about $x = 0$ for $F(x)$. You do not need to simplify any factorials or exponentials that appear in your answer.

Solution: Integrating term by term, and using $F(0) = 9$, we see

$$F(x) = 9 + 4x - \frac{2^2 x^5}{5(2!)} + \frac{2^4 x^9}{9(4!)} - \ldots$$

7. [5 points] Find an expression for the exact value of

$$12 + \frac{4}{5} - \frac{4^2}{2(5)^2} + \frac{4^3}{3(5)^3} + \cdots + \frac{(-1)^{n+1} 4^n}{n 5^n} + \ldots$$

which does not involve an infinite sum (i.e. no sigma notation or "…").

Solution: Using our known Taylor series, we see that this is $12 + \ln\left(1 + \frac{4}{5}\right) = 12 + \ln\left(\frac{9}{5}\right)$.