10. [11 points] Every year, Tommy cooks a feast for his friends and asks them to rate the feast on a scale from 0 to 5. The total average of the ratings from all of his feasts is his *Cumulative Feast Rating.*

After the first feast, Tommy's Cumulative Feast Rating is 4. In the nth year of the feasts, his Cumulative Feast Rating changes by $a_n = 2(-1)^n\left(\frac{1}{2}\right)^n$ from the previous year. That is, if R_n is the Cumulative Feast Rating after the nth feast, $R_1 = 4$ and $R_n = R_{n-1} + a_n$ for $n = 2, 3, 4, \ldots$.

a. [3 points] Tommy wants his feasts to be a success! What will his Cumulative Feast Rating be if he hosts feasts forever? Be sure to show your work.

Solution: The Cumulative Feast Rating tends to

$$4 + \sum_{n=2}^{\infty} 2\left(\frac{-1}{2}\right)^n = 4 + 2\left(\frac{1}{1+\frac{1}{2}} - 1 + \frac{1}{2}\right) = 4 + \frac{1}{3}.$$

b. [3 points] Tommy, knowing he will be unable to supply his friends with feasts forever, so he is interested in how many feasts he will have to host to be within .05 of his final Cumulative Feast Rating. Circle the values of n for which R_n will be within .05 of the value from part (a).

(A) $n = 1$

(B) $n = 3$

$\boxed{\text{(C)}}$ $n = 4$

$\boxed{\text{(D)}}$ $n = 25$

c. [5 points] Suppose that P is the probability density function that Tommy's friends finish the cookies he has prepared for the feast t minutes after he sets them out. Suppose P is given by the following for some $a > 0, b > 0$:

$$P(t) = \begin{cases} bt & 0 < t < a \\ 0 & \text{otherwise} \end{cases}$$

If the median time the cookies take to be finished is 20 minutes, find a and b so that P is a pdf.

Solution: Since P is a pdf, $\int_0^a bt\,dt = 1$. Since the median is 20 minutes, $\int_0^{20} bt\,dt = \frac{1}{2}$, giving

$$\frac{1}{2} = \frac{bt^2}{2}\Big|_0^{20} = 200b \Rightarrow b = \frac{1}{400}.$$

Plugging into the first equation, we get

$$1 = \frac{1}{800}t^2\Big|_0^a = \frac{a^2}{800} \Rightarrow a = \sqrt{800}.$$