4. [6 points] Find the **radius of convergence** of the following power series:

$$\sum_{n=0}^{\infty} \frac{8^n (n!)^3}{(3n)!}(x-5)^{3n}.$$

Show your work including full justifications of any tests you use.

Solution: Setting $a_n = \frac{8^n (n!)^3}{(3n)!}(x-5)^{3n}$, compute

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{8^{n+1}\left((n+1)!\right)^3}{(3n+3)!}\frac{(3n)!}{8^n (n!)^3}|x-5|^3 \\ &= \lim_{n\to\infty} 8\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}|x-5|^3 \\ &= \frac{8}{27}|x-5|^3.\end{aligned}$$

By the ratio test, the power series converges for

$$\frac{8}{27}|x-5|^3 < 1 \iff |x-5| < \left(\frac{27}{8}\right)^{1/3} = \frac{3}{2}.$$

The radius of convergence is $\frac{3}{2}$.