5. [9 points] The radius of convergence of the power series below is 4. Find the **interval of convergence**.

$$\sum_{n=1}^{\infty} \frac{5n}{4^n(n^2+1)}(x+1)^n.$$

Show all your work including full justifications of convergence and divergence of any relevant series. Throughout this problem, you may assume that the radius of convergence is 4 and you do not need to recompute it.

Solution: Observing the power series, the center of the interval of convergence is $x = -1$. Using the radius of convergence, the endpoints are $x = -5$ and $x = 3$.

At $x = -5$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n 5n}{(n^2+1)}.$$

Setting $a_n = \frac{5n}{n^2+1}$, $a_n > 0$, $\lim_{n\to\infty} a_n = 0$, and $a_{n+1} < a_n$. So, by the Alternating Series Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n 5n}{(n^2+1)}$ converges.

At $x = 3$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{5n}{(n^2+1)}.$$

For $n \geq 1$,

$$\frac{5n}{n^2+1} \geq \frac{1}{n},$$

and by the p-test ($p = 1$), $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So, by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{5n}{(n^2+1)}$ diverges.

The interval of convergence is $[-5, 3)$.