5. [9 points] The radius of convergence of the power series below is 4. Find the interval of convergence.

\[ \sum_{n=1}^{\infty} \frac{5n}{4^n(n^2 + 1)}(x + 1)^n. \]

Show all your work including full justifications of convergence and divergence of any relevant series. Throughout this problem, you may assume that the radius of convergence is 4 and you do not need to recompute it.

**Solution:**
Observing the power series, the center of the interval of convergence is \( x = -1 \).
Using the radius of convergence, the endpoints are \( x = -5 \) and \( x = 3 \).

At \( x = -5 \), the power series becomes

\[ \sum_{n=1}^{\infty} \frac{(-1)^n5n}{(n^2 + 1)}. \]

Setting \( a_n = \frac{5n}{n^2 + 1}, \ a_n > 0, \ \lim_{n \to \infty} a_n = 0, \) and \( a_{n+1} < a_n \). So, by the Alternating Series Test, the series \( \sum_{n=1}^{\infty} \frac{(-1)^n5n}{(n^2 + 1)} \) converges.

At \( x = 3 \), the power series becomes

\[ \sum_{n=1}^{\infty} \frac{5n}{(n^2 + 1)}. \]

For \( n \geq 1, \)

\[ \frac{5n}{n^2 + 1} \geq \frac{1}{n}; \]

and by the \( p \)-test \( (p = 1) \), \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. So, by the Direct Comparison Test, \( \sum_{n=1}^{\infty} \frac{5n}{(n^2 + 1)} \) diverges.

The interval of convergence is \( \boxed{[-5, 3)} \).