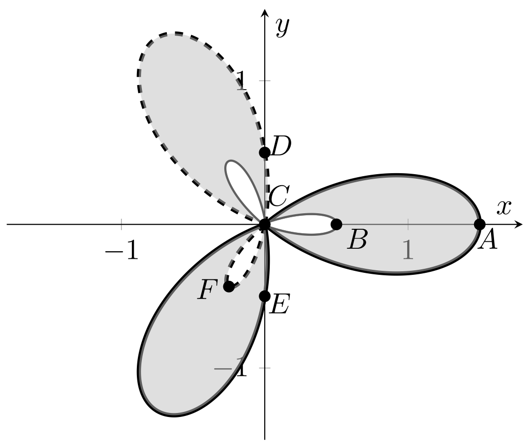
7. [17 points] John is holding a Fan Fair to celebrate the success of his burgeoning fan business. At the fair, John is debuting his new fan, which has blades given by the shaded region of the graph of the polar equation $r = \cos(3\theta) + \frac{1}{2}$ shown below. Note that the graph of $r = \cos(3\theta) + \frac{1}{2}$ is comprised of both the inner and outer loops of the fan blades. One of the activities at the Fan Fair is to guess the perimeter and area of the blades, which can actually be computed explicitly.



a. [4 points] For the values of θ listed below, write on the line the letter of the point corresponding to it.

$\theta = 0$: *A* $\theta = \frac{\pi}{3}$: *F*
$\theta = \frac{\pi}{2}$: *D* $\theta = \pi$: *B*

b. [5 points] Find the 3 values of θ which correspond to the point C (the origin) for $0 \le \theta \le \pi$. Then, determine the interval within $[0, 2\pi]$ for which θ traces out the **dashed** loops in the graph above. (*Hint:* $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$)

Solution: Note that the solutions to $\cos(3\theta) = -\frac{1}{2}$ are, for k an integer

$$3\theta = \frac{2\pi}{3} + 2\pi k$$
$$3\theta = \frac{4\pi}{3} + 2\pi k.$$

So, the first three values are

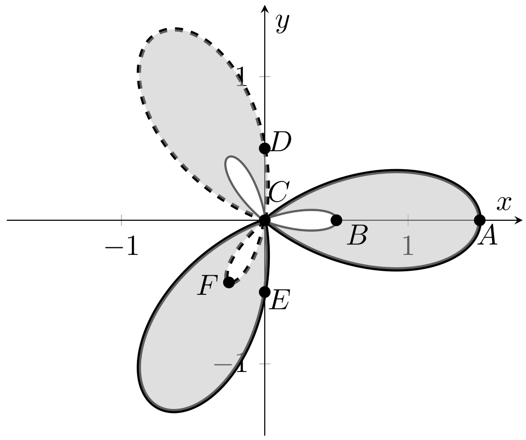
$$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}.$$

Using (a), we know that the dashed loops lie between the first value and the third value when θ corresponds to the point C.

θ = $\frac{2\pi}{9}$, $\frac{4\pi}{9}$, $\frac{8\pi}{9}$

Interval giving θ-values that trace out the dashed loops: $[\frac{2\pi}{9}, \frac{8\pi}{9}]$

7. (continued) Here is a reproduction of the graph from the previous page of the polar equation $r = \cos(3\theta) + \frac{1}{2}$:



c. [4 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total perimeter of the fan blades, including both the inner and outer edges of the fan blades.

Solution: Note that $\frac{dr}{d\theta} = -3\sin(3\theta)$ and the perimeter is given by

$$\int_0^{2\pi} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Total Perimeter = $\int_0^{2\pi} \sqrt{(\cos(3\theta) + \frac{1}{2})^2 + 9\sin^2(3\theta)} d\theta$

d. [4 points] Write, but do not evaluate, an expression giving the total area of all 3 fan blades (the shaded region of the graph). (*Hint*: Your answer from (b) may be handy, but is not strictly necessary)

Solution: Using (b), the small dashed loop is traced out for $\frac{2\pi}{9} \le \theta \le \frac{4\pi}{9}$ and the large dashed loop is traced out for $\frac{4\pi}{9} \le \theta \le \frac{8\pi}{9}$, so the area of one small loop is

$$\int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta$$

and the area of one large loop is

$$\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta.$$

Exploiting the symmetry of the fan, we get the total area below.

Total Area = $3\left[\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta - \int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta\right]$