8. [16 points] Molly and Erin are two bumper car enthusiasts who hate bumping into things. So, they get on the bumper cars, ride until they bump into each other, and then stop riding the bumper cars. At time t minutes after they start driving their bumper cars, Molly's position is given by

$$M(t) = \left(4 - 6\cos t, 3t + \pi^2\right)$$

and Erin's position is given by

$$E(t) = \left(2\cos t, (t-2\pi)^2 + 3t + \frac{8\pi^2}{9}\right),$$

where all distances are in meters.

a. [4 points] How long do they ride the bumper cars? Make sure to include units.

Solution: We set

$$4 - 6\cos t = 2\cos t \quad \text{and} \quad 3t + \pi^2 = (t-2\pi)^2 + 3t + \frac{8\pi^2}{9},$$

we get

$$\cos t = \frac{1}{2} \quad \text{and} \quad (t-2\pi)^2 = \frac{\pi^2}{9},$$

so the x-coordinates are equal when $t = \frac{\pi}{3}, \frac{5\pi}{3} + 2\pi k$ and the y-coordinates are equal when $t = \frac{5\pi}{3}, \frac{7\pi}{3}$. So, they ride the bumper cars for $\frac{5\pi}{3}$ minutes.

b. [4 points] Find an explicit expression for Erin's speed t minutes after she starts driving her bumper cars, before the collision (your expression should not contain any integrals nor the letters M, E). Make sure to include units.

Solution: For Erin, $\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = 2(t-2\pi) + 3$. So, her speed is

$$\sqrt{4\sin^2 t + (2(t-2\pi)+3)^2} \text{ meters/minute}$$

c. [4 points] Write, but do not evaluate, an expression involving integrals that gives the total distance that Erin travelled before the collision. Make sure to include units.

Solution: Using the answer to (a),

$$\int_0^{\frac{5\pi}{3}} \sqrt{4\sin^2 t + (2(t-2\pi)+3)^2}\,dt \text{ meters}$$

d. [4 points] Suppose the positive y-direction in the xy-plane is North. At $t = 0$, Molly is facing directly North. Find all other times $t > 0$ (if any) after they start riding their bumper cars, but before their collision, when Molly is facing directly North. Make sure to include units.

Solution: For, Molly $\frac{dx}{dt} = 6\sin t$ and $\frac{dy}{dt} = 3$. So, $\frac{dx}{dt} = 0$ when $t = k\pi$. The only $t > 0$ before the collision where $\frac{dx}{dt} = 0$ is π minutes.