1. [11 points] Nancy is on an airplane traveling to see Carlos, and the flight is delayed. The function $f(t)$ is the probability density function (pdf) for how many hours, t, the flight will be delayed. The function $f(t)$ is given by

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ -\frac{3}{4}(t-1)^2 + \frac{3}{4} & \text{if } 0 < t < 2, \\ 0 & \text{if } t \geq 2 \end{cases}$$

a. [6 points] Write a formula for the cumulative distribution function (cdf) $F(t)$ corresponding to the pdf $f(t)$. Your answer should not involve any integrals or the letter $f(t)$. Write your answer using the partially given piecewise notation below.

Solution: The cdf is given by

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f(s)\, ds \\ &= \int_{0}^{t} f(s)\, ds. \end{aligned}$$

The second equality is true because $f(t) = 0$ for $t \leq 0$. This also means that $F(t) = 0$ for $t \leq 0$. If $0 < t < 2$, then we have:

$$\begin{aligned} F(t) &= \int_{0}^{t} f(s)\, ds \\ &= \int_{0}^{t} \Big(-\frac{3}{4}(s-1)^2 + \frac{3}{4} \Big)\, ds \\ &= \Big(-\frac{1}{4}(s-1)^3 + \frac{3}{4}s \Big)\Big|_{0}^{t} \\ &= -\frac{1}{4}(t-1)^3 - \frac{1}{4} + \frac{3}{4}t. \end{aligned}$$

This gives $F(t)$ for $0 < t < 2$. For $t \geq 2$, note that $f(t) = 0$ for all such t, and therefore $F(t)$ must be constant on $(2, \infty)$. Since cdfs must tend to 1 as the input tends to ∞, this means $F(t) = 1$ for $t \geq 2$. We can also see this by computing: if $t \geq 2$,

$$\begin{aligned} F(t) &= \int_{0}^{t} f(s)\, ds \\ &= \int_{0}^{2} f(s)\, ds + \int_{2}^{t} f(s)\, ds \\ &= \int_{0}^{2} \Big(-\frac{3}{4}(s-1)^2 + \frac{3}{4} \Big)\, ds \\ &= \Big(-\frac{1}{4}(s-1)^3 + \frac{3}{4}s \Big)\Big|_{0}^{2} \\ &= -\frac{1}{4} + \frac{3}{2} - \frac{1}{4} = 1. \end{aligned}$$

Below we write down the formula for $F(t)$ using the provided piecewise notation.

$$F(t) = \begin{cases} 0 & \textbf{if } t \leq 0 \\ -\frac{1}{4}(t-1)^3 + \frac{3}{4}t - \frac{1}{4} & \textbf{if } 0 < t < 2 \\ 1 & \textbf{if } t \geq 2 \end{cases}$$

b. [2 points] What is the probability that Nancy's flight will be delayed less than 30 minutes?

Solution: This is equal $F(1/2)$. We can use our answer to part (a) to get

$$F(1/2) = -\frac{1}{4}(-1/2)^3 + \frac{3}{4}\cdot\frac{1}{2} - \frac{1}{4} = \frac{5}{32}.$$

c. [3 points] Carlos wants to find the mean amount of time the flight will be delayed, so he can arrive at the airport at the right time. Write an explicit expression involving integrals that gives the mean amount of time the flight will be delayed. Do not evaluate your expression. Your answer should not contain the letter f.

Solution: The mean is:

$$\int_{-\infty}^{\infty} tf(t)\,dt = \int_0^2 \left(-\frac{3}{4}t(t-1)^2 + \frac{3}{4}t\right) dt$$