3. [13 points] A function \( g(x) \) has Taylor series centered at \( x = 5 \) given by
\[
\sum_{n=0}^{\infty} \frac{(-1)^n(x - 5)^{n+1}}{(n + 1) \cdot 4^n}.
\]

a. [2 points] Is \( g(x) \) increasing or decreasing near \( x = 5 \)? Briefly justify your answer.

\[
\text{Solution: 
}\frac{g'(5)}{g'(5)} = \frac{(-1)^0}{(0+1)\cdot 4^0} = 1 > 0, \text{ so } g(x) \text{ is increasing near } x = 5.
\]

b. [3 points] Find \( g^{(1001)}(5) \).

\[
\text{Solution: 
}\frac{g^{(1001)}(5)}{g^{(1001)}(5)} = \frac{(-1)^{1000}}{(1000+1)\cdot 4^{1000}}.
\]

\[
g^{(1001)}(5) = \frac{1001!}{1001\cdot 4^{1000}}.
\]
c. [8 points] Given that the radius of convergence of this Taylor series is 4 (do NOT show this), find the interval of convergence of this Taylor series. Show all your work, including full justification for series behavior.

**Solution:** Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are 5 ± 4 = 1, 9. At \( x = 1 \), the series is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(-4)^{n+1}}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{4(-1)^n(-1)^{n+1}}{n + 1} = -4 \sum_{n=0}^{\infty} \frac{1}{n+1}.
\]

To determine the behavior of this, we use the limit comparison test with comparison series \( \sum_{n=1}^{\infty} \frac{1}{n} \). We have (with \( a_n = \frac{1}{n+1} \) and \( b_n = \frac{1}{n} \)):

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{n + 1} = 1.
\]

Since 1 is a positive number, and since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges by the \( p \)-series test with \( p = 1 \), the limit comparison tells us that the series \( \sum_{n=0}^{\infty} \frac{1}{n+1} \) diverges. Therefore \( -4 \sum_{n=0}^{\infty} \frac{1}{n+1} \) diverges. So \( x = 1 \) is not included in the interval of convergence.

At \( x = 9 \), the series is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n4^{n+1}}{(n+1)4^n} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1}.
\]

Since \( a_n = \frac{1}{n+1} \to 0 \) as \( n \to \infty \), and \( a_n \) is decreasing, by the alternating series test this series converges.

Interval of convergence: \((1, 9]\)