3. [13 points] A function $g(x)$ has Taylor series centered at $x = 5$ given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^{n+1}}{(n+1) \cdot 4^n}.$$

a. [2 points] Is $g(x)$ increasing or decreasing near $x = 5$? Briefly justify your answer.

Solution: The coefficient of $(x-5)$ in the Taylor series is equal to $g'(5)$. Therefore $g'(5) = \frac{(-1)^0}{(0+1)\cdot 4^0} = 1 > 0$, so $g(x)$ is increasing near $x = 5$.

b. [3 points] Find $g^{(1001)}(5)$.

Solution: The coefficient of $(x-5)^n$ in the Taylor series for $g(x)$ is $\frac{g^{(n)}(5)}{n!}$. We see that the exponent on $(x-5)$ is 1001 when $n = 1000$. Therefore $g^{(1001)}(5)$ is equal to 1001! times $\frac{(-1)^{1000}}{(1000+1)\cdot 4^{1000}}$.

$g^{(1001)}(5)=$ $\frac{1001!}{1001 \cdot 4^{1000}}$

c. [8 points] Given that the radius of convergence of this Taylor series is 4 (do NOT show this), find the **interval** of convergence of this Taylor series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $5 \pm 4 = 1, 9$. At $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-4)^{n+1}}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{4(-1)^n(-1)^{n+1}}{n+1} = -4\sum_{n=0}^{\infty} \frac{1}{n+1}.$$

To determine the behavior of this, we use the limit comparison test with comparison series $\sum_{n=1}^{\infty} \frac{1}{n}$. We have (with $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$):

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since 1 is a positive number, and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test with $p = 1$, the limit comparison tells us that the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. Therefore $-4\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. So $x = 1$ is not included in the interval of convergence.

At $x = 9$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^{n+1}}{(n+1)4^n} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

Since $a_n = \frac{1}{n+1} \to 0$ as $n \to \infty$, and a_n is decreasing, by the alternating series test this series converges.

Interval of convergence: $(1, 9]$