7. [12 points] The rate of vertical growth $r(t)$ of a tree, in meters per **month**, is given by

$$r(t) = \frac{10}{(t+1)^{3/2}}.$$

Here, t is measured in **months** after the tree was planted. **When the tree was planted its height was 1 meter.**

a. [4 points] Write an expression, possibly involving one or more integrals, for the height of the tree after exactly 1 **year** has passed since planting it. You do not need to evaluate your integral(s).

Solution: Let $h(t)$ be the height of the tree, in meters, t months after it was planted. So we are looking for $h(12)$. By the fundamental theorem of calculus,

$$h(12) - h(0) = \int_0^{12} r(t)\, dt.$$

Since at the time it was planted ($t = 0$), the tree was 1 meter tall, $h(0) = 1$. Therefore

$$h(12) = 1 + \int_0^{12} \frac{10}{(t+1)^{3/2}}\, dt.$$

b. [2 points] Let $h(t)$ be the height of the tree, in meters, t **months** after it was planted. Write an expression, possibly involving one or more integrals, for the function $h(t)$. You do not need to evaluate your integral(s).

Solution: By the FTC,

$$h(t) = 1 + \int_0^{t} \frac{10}{(s+1)^{3/2}}\, ds.$$

c. [6 points] Assuming the tree lives long enough, will the tree ever grow more than 20 meters tall? Justify your answer, and be sure to use proper notation.

Solution: The long-term height of the tree, as $t \to \infty$, is

$$\begin{aligned}
\lim_{t\to\infty} h(t) &= 1 + \lim_{t\to\infty} \int_0^t \frac{10}{(s+1)^{3/2}}\, ds \\
&= 1 + \lim_{t\to\infty} -20(s+1)^{-1/2}\Big|_0^t \\
&= 1 - 20 \lim_{t\to\infty} \left(\frac{1}{\sqrt{1+t}} - 1 \right) \\
&= 21.
\end{aligned}$$

Therefore at some point, the tree *will* grow more than 20 meters tall.