6. [10 points] Consider an infinitely differentiable function $f(x)$. The following table gives some values of $f(x)$ and its derivatives at $x = 1$:

$f(1)$	$f'(1)$	$f''(1)$	$f'''(1)$
$\pi/4$	$1/2$	$-1/4$	2

a. [4 points] Write down $P_3(x)$, the third-degree Taylor polynomial of $f(x)$ about $x = 1$. You do not need to simplify.

$P_3(x) =$ $\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4 \cdot 2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$

b. [3 points] Recall that $f(x) \approx P_3(x)$ near $x = 1$. Use this and the fact that $f(1.5) = \pi/3$ to write an approximation for π. You do not need to simplify your answer. Your answer should not contain the symbol π.

Solution: Using $f(x) \approx P_3(x)$ near $x = 1$, we have

$$\frac{\pi}{3} = f(1.5) \approx P_3(1.5) = \frac{\pi}{4} + \frac{1}{2}(0.5) - \frac{1}{4 \cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3.$$

So $\frac{\pi}{3} - \frac{\pi}{4} \approx \frac{1}{2}(0.5) - \frac{1}{4 \cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3$. By getting a common denominator of 12 on the left-hand side of the approximation, we get

$$\pi \approx 12 \cdot \left(\frac{1}{2}(0.5) - \frac{1}{4 \cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3\right).$$

$\pi \approx$ $12 \cdot \left(\frac{1}{2}(0.5) - \frac{1}{4 \cdot 2!}(0.5)^2 + \frac{2}{3!}(0.5)^3\right)$

c. [3 points]
Use the Taylor polynomial from part **a.** to approximate the definite integral

$$\int_1^{1.1} f(x)\, dx.$$

You do not need to simplify your answer.

Solution:

$$\int_1^{1.1} f(x)\, dx \approx \int_1^{1.1} P_3(x)\, dx = \frac{\pi}{4}(0.1) + \frac{1}{2 \cdot 2}(0.1)^2 - \frac{1}{3 \cdot 4 \cdot 2!}(0.1)^3 + \frac{2}{4 \cdot 3!}(0.1)^4.$$

Answer: $\frac{\pi}{4}(0.1) + \frac{1}{2 \cdot 2}(0.1)^2 - \frac{1}{3 \cdot 4 \cdot 2!}(0.1)^3 + \frac{2}{4 \cdot 3!}(0.1)^4$