8. [9 points] Gabriella is developing a new kind of vuvuzela. In order to come up with a new method, she first considers the old way she made her instruments.

a. [4 points] Gabriella initially made her vuvuzelas by considering a positive function $f(x)$, and forming a region $\mathcal{R}$ between $y = f(x)$ and the x-axis on the interval $[2, \infty)$. She rotated $\mathcal{R}$ about the x-axis to form the shape of the vuvuzela. Write an integral which gives the volume of the vuvuzela. Your answer will involve the function $f(x)$.

Answer: $\int_2^\infty \pi(f(x))^2 \, \mathrm{d}x$

b. [5 points] For her new batch of vuvuzelas, Gabriella considers an entirely different shape. The volume of the new design of vuvuzela is given by

$$\int_2^\infty \frac{x}{(x^2+5)^2} \, \mathrm{d}x.$$

Compute the value of this integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

Solution: We begin by writing the improper integral as a corresponding limit. Using a substitution of $u = x^2 + 5$, we have

$$\begin{aligned}\int_2^\infty \frac{x}{(x^2+5)^2} \, \mathrm{d}x &= \lim_{b\to\infty} \int_2^b \frac{x}{(x^2+5)^2} \, \mathrm{d}x \\ &= \lim_{b\to\infty} \int_9^{b^2+5} \frac{(1/2)}{u^2} \, \mathrm{d}u \\ &= \lim_{b\to\infty} -\frac{1}{2(b^2+5)} + \frac{1}{2(9)} \\ &= -0 + \frac{1}{18}.\end{aligned}$$

Therefore the integral converges to $\frac{1}{18}$.

Circle one: **Diverges** **Converges to** $\frac{1}{18}$