9. [8 points] A power series centered at $x = 3$ given by

$$\sum_{n=0}^{\infty} \frac{n^2+1}{2^n(n+2)}(x-3)^n.$$

The radius of convergence of this power series is 2 (do NOT show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $3 \pm 2 = 1, 5$. At $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n^2+1}{2^n(n+2)}(1-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n^2+1)}{n+2}.$$

To determine the behavior of this, we use the nth term test for divergence.

$$\lim_{n\to\infty} \frac{(-1)^n(n^2+1)}{n+2} = \lim_{n\to\infty} (-1)^n n.$$

which does not exist, and so the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n^2+1)}{n+2}$ diverges by the nth term test for divergence. So $x = 1$ is not included in the interval of convergence.

Similarly, at $x = 5$, the series is

$$\sum_{n=0}^{\infty} \frac{n^2+1}{2^n(n+2)}(5-3)^n = \sum_{n=0}^{\infty} \frac{n^2+1}{n+2}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{n+2} = \infty$, this series also diverges by the nth term test for divergence. Therefore, the interval of convergence is $(1, 5)$.

Interval of convergence: $(1, 5)$