2. [10 points] A power series centered at $x = 4$ is given by

$$\sum_{n=1}^{\infty} \frac{n+1}{11^n \cdot n^{3/2}} (x-4)^n.$$

The radius of convergence of this power series is 11 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $4 - 11 = -7$, and $4 + 11 = 15$.
At $x = -7$, the series is

$$\sum_{n=1}^{\infty} \frac{n+1}{11^n \cdot n^{3/2}} (-11)^n = \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^{3/2}}.$$

To determine the behavior of this, we use the Alternating Series Test.

$$\lim_{n\to\infty} \frac{n+1}{n^{3/2}} = \lim_{n\to\infty} \frac{n}{n^{3/2}} = \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0.$$

and for all $n \geq 1$,

$$0 < \frac{n+2}{(n+1)^{3/2}} < \frac{n+1}{n^{3/2}},$$

so by the Alternating Series test, $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^{3/2}}$ converges.
Therefore $x = -7$ is included in the interval of convergence.

At $x = 15$, the series is

$$\sum_{n=1}^{\infty} \frac{n+1}{11^n \cdot n^{3/2}} (11)^n = \sum_{n=1}^{\infty} \frac{n+1}{n^{3/2}}.$$

We can show that this series diverges in a few different ways - for example, using the Limit Comparison Test or the Integral Test - but we will use the (Direct) Comparison Test.
For $n \geq 1$, $\frac{n+1}{n^{3/2}} > \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}$, and $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, by the p-test, with $p = \frac{1}{2}$. Therefore, by the (Direct) Comparison Test, $\sum_{n=1}^{\infty} \frac{n+1}{n^{3/2}}$ diverges. This tells us that $x = 15$ is not included in the interval of convergence.
Therefore, the interval of convergence is $[-7, 15)$.

Answer: Interval of convergence: $[-7, 15)$