6. [5 points] Consider the function $F(x)$ defined by its Taylor series around $x = 0$,

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)!}.$$

Find $F^{(2024)}(0)$ and $F^{(2025)}(0)$. You do not need to simplify your answers.

Solution: All even powers of x have zero coefficient. Hence $F^{(2024)}(0) = 0$.
On the other hand, $x^{(2025)}$ appears when $2n + 1 = 2025$, i.e., when $n = 1012$. Thus

$$\frac{F^{(2025)}(0)}{2025!} = \frac{(-1)^{1012}}{(1012!)(2025!)}$$

Rearranging and simplifying, we get

$$F^{(2025)}(0) = \frac{1}{1012!}$$

Answer: $F^{(2024)}(0) =$ 0 and $F^{(2025)}(0) =$ $\frac{1}{1012!}$

7. [7 points] Consider the function

$$g(x) = \frac{3}{\sqrt{1 + 5x^2}}.$$

a. [5 points] Give the first three nonzero terms of the Taylor series of $g(x)$ centered about $x = 0$. Show all your work.

Solution: Using the known Taylor series for $(1 + y)^p$ with $y = 5x^2$ and $p = -\frac{1}{2}$ we get that the Taylor series for $g(x)$ centered about $x = 0$ is:

$$3\left(1 - \frac{1}{2}(5x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(5x^2)^2 + \cdots\right) = 3\left(1 - \frac{5}{2}x^2 + \frac{75}{8}x^4 + \cdots\right)$$

Answer: $3\left(1 - \frac{5}{2}x^2 + \frac{75}{8}x^4 + \cdots\right)$

b. [2 points] What is the radius of convergence of the Taylor series for $g(x)$?

Solution: The interval of convergence for $(1 + y)^p$ is $-1 < y < 1$. Hence the interval of convergence for the Taylor series of $g(x)$ centered at $x = 0$ is $-1 < 5x^2 < 1$, i.e. $-\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$.
Therefore the radius of convergence is $\frac{1}{\sqrt{5}}$.

Answer: $\frac{1}{\sqrt{5}}$