9. [6 points] Let $f(x)$ and $g(x)$ be two continuous and differentiable functions on $[1, \infty)$. Further, suppose these functions have the following properties:

- $F(x) = x(g(x) + 1)$ is an antiderivative of $f(x)$ for $x \geq 1$,
- $g(1) = 10$,
- $\lim_{x\to\infty} g(x) = -1$,
- $\lim_{x\to\infty} x^2 g'(x) = 17$.

Compute the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_1^\infty f(x)\,\mathrm{d}x$$

Circle one: **Diverges** **Converges to** -28

Solution: We start by rewriting this improper integral as a limit, and then use the First Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{b\to\infty} \int_1^b f(x)\,\mathrm{d}x \\
&= \lim_{b\to\infty} F(b) - F(1) \\
&= \lim_{b\to\infty} b(g(b)+1) - (1)(g(1)+1) \\
&= \lim_{b\to\infty} b(g(b)+1) - 11 \\
&= \lim_{b\to\infty} \frac{g(b)+1}{1/b} - 11
\end{aligned}$$

As $\lim_{b\to\infty} g(b) + 1 = \lim_{b\to\infty} 1/b = 0$, we try to use L'Hôpital's Rule. We obtain:

$$\begin{aligned}
\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{b\to\infty} -\frac{g'(b)}{1/b^2} - 11 \\
&= \lim_{b\to\infty} -b^2 g'(b) - 11 \\
&= -17 - 11 \\
&= -28
\end{aligned}$$

Therefore, the integral converges to -28.