2. [14 points] A function $f(x)$ is defined on the interval $(0,2)$ by its Taylor series around $x=1$,

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{2n+1}(x-1)^{2n+1}.$$

a. [3 points] Find $f^{(2025)}(1)$ and $f^{(2026)}(1)$.

Solution: If we write down the Taylor series of f around 1, $f^{(2025)}(1)$ appears in the coefficient of $(x-1)^{2025}$, where $n=1012$. Therefore, comparing coefficients, we get

$$\frac{1}{2025}=\frac{f^{(2025)}(1)}{2025!},$$

and thus $f^{(2025)}(1)=\dfrac{2025!}{2025}=2024!$.
We also notice that all even powers of x have zero coefficient, so $f^{(2026)}(1)=0$.

Answer: $f^{(2025)}(1)=$ 2024! and $f^{(2026)}(1)=$ 0

b. [5 points] Find the degree 7 Taylor polynomial around $x=1$ for $f(x)$.

Solution: We include all the terms up to $(x-1)^7$. Therefore, the degree 7 Taylor polynomial is

$$(x-1)+\frac{1}{3}(x-1)^3+\frac{1}{5}(x-1)^5+\frac{1}{7}(x-1)^7.$$

Answer: $(x-1)+\frac{1}{3}(x-1)^3+\frac{1}{5}(x-1)^5+\frac{1}{7}(x-1)^7$

c. [3 points] Find the degree 6 Taylor polynomial around $x=1$ for $f'(x)$, the **derivative** of $f(x)$.

Solution: We take the derivative of the polynomial in **b.** and get

$$1+(x-1)^2+(x-1)^4+(x-1)^6.$$

Answer: $1+(x-1)^2+(x-1)^4+(x-1)^6$

d. [3 points] Find a closed-form expression of the derivative $f'(x)$ which applies on the interval $(0,2)$. Closed form means your answer should not include ellipses or sigma notation.

Solution: Using our known Taylor series, we notice that

$$f'(x)=\sum_{n=0}^{\infty}(x-1)^{2n}=\frac{1}{1-(x-1)^2}$$

on $(0,2)$, which is the interval of convergence of $f(x)$.

Answer: $f'(x)=\dfrac{1}{1-(x-1)^2}$