5. [10 points] Consider the Taylor series, $\sum_{n=1}^{\infty} \frac{9n^2+8}{2^n \cdot (n^3+1)} (x+3)^n$.

a. [1 point] Determine the center, a, of the Taylor series.

Answer: $a =$ -3

b. [9 points] The radius of convergence of the Taylor series is 2 (you do **not** need to show this). Determine the interval of convergence of the Taylor series. Show all your work, including full justification for series behavior.

Solution: Given that the center is $x = -3$ and that the radius of convergence is 2, we need to check the endpoints $x = -1$ and $x = -5$.
When $x = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{9n^2+8}{2^n \cdot (n^3+1)} 2^n = \sum_{n=1}^{\infty} \frac{9n^2+8}{n^3+1}.$$

We can use the Limit Comparison Test (LCT) or Direct Comparison Test (DCT) to show divergence. For the Limit Comparison Test (LCT), we note that

$$\lim_{x\to\infty} \frac{\frac{9n^2+8}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{9n^3+8n}{n^3+1} = 9 > 0,$$

and that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-test with $p = 1$). Therefore, by the LCT, $\sum_{n=1}^{\infty} \frac{9n^2+8}{n^3+1}$ diverges.
Alternatively, for the Direct Comparison Test (DCT), we note that for $n \geq 1$,

$$\frac{9n^2+8}{n^3+1} > \frac{9n^2}{n^3+1} \geq \frac{9n^2}{2n^3} = \frac{9}{2n},$$

and that $\sum_{n=1}^{\infty} \frac{9}{2n}$ diverges (by the p-test with $p = 1$). Therefore, by the DCT, $\sum_{n=1}^{\infty} \frac{9n^2+8}{n^3+1}$ diverges.
When $x = -5$, the Taylor series is

$$\sum_{n=1}^{\infty} \frac{9n^2+8}{2^n \cdot (n^3+1)} (-2)^n = \sum_{n=1}^{\infty} (-1)^n \frac{9n^2+8}{n^3+1}.$$

We want to use Alternating Series Test (AST) to justify convergence. We see that

1. $\lim_{n\to\infty} \frac{9n^2+8}{n^3+1} = 0$, and
2. $\frac{9n^2+8}{n^3+1}$ is monotone decreasing for $n \geq 1$.

so the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{9n^2+8}{n^3+1}$$

converges by alternating series test (AST). Hence the interval of convergence is $[-5, -1)$.

Answer: The interval of convergence is $[-5, -1)$